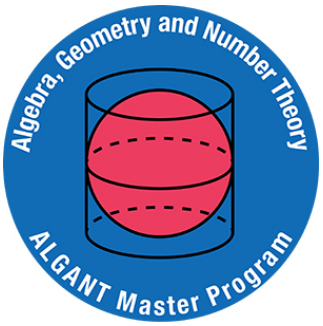


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Lubin-Tate towers

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Abstract

Let K be a non-Archimedean local field with ring of integers $\mathcal{O}_K$ and residue field k. In this master thesis, after introducing the necessary concepts, Lubin-Tate spaces of arbitrary level associated to a formal 1-dimensional $\mathcal{O}_K$-module over k of height $h < \infty$ are defined and proven to be representable, as in [Dri74]. Together they form the Lubin-Tate tower and we show, à la [Wei10], how we can make sense of this tower as the "infinite level Lubin-Tate space".
Throughout the text many details will be given, which are not always found in the literature.

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Introduction

Let K be a non-Archimedean local field, with ring of integers $\mathcal{O}_K$, uniformizer π, and residue field k of characteristic p and cardinality q.

This is a (master) thesis about Lubin-Tate spaces, which means a great part of it is spent treating formal groups and their generalization: formal $\mathcal{O}_K$-modules. Formal groups (over a ring R) can be thought of as "groups without elements" for they only come with a formal power series $S(X,Y)$ (with coefficients in R) that should be thought of as the sum of X and Y, and as such satisfies standard group axioms like associativity $S(S(X,Y),Z) = S(X,S(Y,Z))$. This power series is called a formal group law and should be thought of as a rule of how one ought to add elements, if there were elements... A more highbrow explanation is that formal groups (over R) are certain group objects in the category of formal schemes over R.
Formal $\mathcal{O}_K$-modules (over an $\mathcal{O}_K$-algebra R) are similar, but now there are also power series $[a](X)$ for every $a \in \mathcal{O}_K$ expressing scalar multiplication and satisying some conditions. They can be seen as $\mathcal{O}_K$-module objects in the category of formal schemes over R.

In one of the simplest cases, when $K = \mathbb{Q}_p$, the (level 0) Lubin-Tate space is an affine formal scheme over $\mathbb{Z}_p$ parametrizing lifts – usually called deformations – of a certain formal group $\overline{F}(X,Y)$ defined over $\mathbb{F}_p$. For example, its $\mathbb{Z}_p$-points are (isomorphism classes of) deformations $F(X,Y)$ of $\overline{F}(X,Y)$ over $\mathbb{Z}_p$. That is, $F(X,Y)$ is a formal group law with coefficients in $\mathbb{Z}_p$ that reduces to $\overline{F}(X,Y)$ modulo p.
Lubin and Tate were the first to study this situation [LT66] and from their work one sees that the space equals the formal spectrum

$$\operatorname{Spf} \mathbb{Z}_p[\![X_1, \ldots, X_{h-1}]\!],$$

where h is the height of $\overline{F}$, an important invariant which says that the scalar multiplication by p power series $[p](X)$ of $\overline{F}$ is of the form

$$[p](X) = \text{unit} \cdot X^{p^h} + \text{higher order terms}.$$

In fact, the isomorphism class of a lift $F(X,Y)$ of $\overline{F}(X,Y)$ turns out to only depend on the coefficients of $X^p, X^{p^2}, \ldots, X^{p^{h-1}}$ in the corresponding lift of $[p](X)$. This insight is key in computing the Lubin-Tate space.

Drinfeld generalized the work of Lubin and Tate to the general case of formal $\mathcal{O}_K$-modules in [Dri74]. In this case, the (level 0) Lubin-Tate space is an affine

formal $\mathcal{O}_K$-scheme parametrizing the isomorphism classes of deformations of a height h formal $\mathcal{O}_K$-module $\overline{F}$ defined over k and is isomorphic to

$$\operatorname{Spf} \mathcal{O}_K[\![X_1, \ldots, X_{h-1}]\!].$$

In this case, height h means that the scalar multiplication by the uniformizer π in the formal $\mathcal{O}_K$-module $\overline{F}$ looks like

$$[\pi](X) = \text{unit} \cdot X^{q^h} + \text{higher order terms}.$$

Again, it is the case that the class of a lift only depends on the coefficients $X^q, X^{q^2}, \ldots, X^{q^{h-1}}$ of the corresponding lift of $[\pi](X)$. Ones interest in $[\pi](X)$ goes even further, as we will soon see.
This Lubin-Tate space is usually denoted by $\mathcal{M}_0$ or $\mathcal{M}_{\overline{F},0}$. This stresses that it is the zeroth level. The other levels were defined by Drinfeld in the same paper as mentioned before: the n-th level Lubin-Tate space $\mathcal{M}_n$ is the affine formal $\mathcal{O}_K$-scheme parametrizing classes of lifts F of $\overline{F}$ together with a Drinfeld level π^n structure on F – this is a structure on the π^n-torsion parts of the lift F, which is of a slightly technical nature.
It turns out that there are natural morphisms

$$\mathcal{M}_{n+1} \to \mathcal{M}_n$$

for every $n \geq 0$. This gives us the Lubin-Tate tower:

$$\cdots \to \mathcal{M}_3 \to \mathcal{M}_2 \to \mathcal{M}_1 \to \mathcal{M}_0.$$

The aim of this thesis is to prove all of the above and show that the Lubin-Tate tower "as a whole" is of a perfectoid nature, in some sense.

Why study Lubin-Tate spaces?

Let us now briefly explain why this might be interesting. For this, we first go back to the specific case of $K = \mathbb{Q}_p$ and $h = 1$. Then the 0-th level Lubin-Tate space $\operatorname{Spf} \mathbb{Z}_p$ could not be more boring. It being a point expresses the fact that there is, up to isomorphism, only one deformation of the height 1 formal group $\overline{F}$. An example of a possible deformation F that might occur (if $\overline{F}$ is compatibly chosen as its reduction modulo p) is one whose p^n-scalar multiplication power series look like

$$p_F^n(X) \coloneqq (X+1)^{p^n} - 1.$$

(Check that the reduction modulo p of $p_F^n(X)$ is X^p, so that $\overline{F}$ is indeed of height 1.)
The formal scheme $\mathcal{M}_n$ parametrizing *the* (class of the) lift together with some structure on its p^n-torsion, turn out to be actually only parametrizing the roots of p_F^n. That is,

$$\mathcal{M}_n = \operatorname{Spf} \mathcal{O}_{K_n},$$

where K_n is the field obtained from $\mathbb{Q}_p$ by adding all roots of $p_F^n(X) = (X+1)^{p^n} - 1$. Thus, writing ζ_{p^n} for a primitive p^n-th root of unity, the level n Lubin-Tate space is nothing else than

$$\operatorname{Spf} \mathbb{Z}_p[\zeta_{p^n}].$$

On the level of rings, the associated Lubin-Tate tower becomes

$$\mathbb{Z}_p \hookrightarrow \mathbb{Z}_p[\zeta_p] \hookrightarrow \mathbb{Z}_p[\zeta_{p^2}] \hookrightarrow \cdots .$$

So "considering it as a whole" it is $\mathbb{Z}_p[\zeta_{p^\infty}]$, the ring of integers of $\mathbb{Q}_p(\zeta_{p^\infty})$. The reader familiar with the (local) Kronecker-Weber theorem will recognize that the maximal abelian extension of $\mathbb{Q}_p$ decomposes into the maximal unramfied extension $\mathbb{Q}_p^{\mathrm{unr}}$ and the field just constructed, that is

$$\mathbb{Q}_p^{\mathrm{ab}} = \mathbb{Q}_p(\zeta_{p^\infty})\, \mathbb{Q}_p^{\mathrm{unr}} .$$

On each $\mathbb{Q}_p(\zeta_{p^n})$ there is an action of the automorphism group of $\overline{F}$, which is $\mathcal{O}_K^\times$, that leaves $\mathbb{Q}_p$ fixed and in fact, this action identifies $\operatorname{Gal}\mathbb{Q}_p(\zeta_{p^n})/\mathbb{Q}_p$ with the quotient $\mathbb{Z}_p^\times/(1+p^n\mathbb{Z}_p)$. These and other insights about this Lubin-Tate tower lead to proves of local class field theory. Using a height 1 formal $\mathcal{O}_K$-module $\overline{F}$ over k, the can be done for arbitrary K and this was discovered by Lubin and Tate [LT65], whence it is often referred to as the Lubin-Tate approach to local class field theory.

Now, as we have explained before, this Lubin-Tate tower can also be defined for $\overline{F}$ of height not 1 (to be precise: the height may also not be infinite). One might wonder if this setting in some way generalizes local class field theory. It actually does, in what is generally seen as the generalization of local class field theory: the local Langlands program. Let us be a bit more precise. In general, the Lubin-Tate tower admits an action of

$$\mathrm{GL}_h(K) \times D^\times \times W_K,$$

where $D^\times$ are the units of the central division algebra of invariant $1/h$ over K and $W_K \subset \operatorname{Gal}(\overline{K}/K)$ is the Weil group of K. It turns out that using these actions, the ℓ-adic étale cohomology of the Lubin-Tate tower "realizes the Jacquet-Langlands correspondence (between $\mathrm{GL}_h(K)$ and $D^\times$) and the local Langlands correspondence (between $\mathrm{GL}_h(K)$ and W_K)" ([Wei10]). In the case $h = 1$ this is local class field theory. See the introduction of [Wei10] for some details and especially references.

We want to stress to the reader that this thesis does *not* discuss the local Langlands program or even local class field theory! We merely mention it as an advertisement of what can be studied after reading this text.

In the spirit of advertising, we would finally like to mention one more thing: we only mentioned the correspondences realized via the ℓ-adic (where we understood: $\ell \neq p$). Indeed, the p-adic case is still mysterious and the local Langlands and Jacquet-Langlands correspondences have not been established in these cases. Whatever the reason for this is, a relatively recent paper of Scholze [Sch15] he defines functors that are both candidates for the p-adic local Langlands correspondence and for the p-adic Jacquet-Langlands correspondence (when K is a p-adic field). This construction depends on the p-adic cohomology of, again, the Lubin-Tate tower with its action of the triple product group as described above.

In order to see whether Scholze's candidate functors do the job, they need to be studied in more detail. With this we hope to provide more motivation for the study of Lubin-Tate towers, as it shows that they are still very relevant in modern research. Finally, let us note that Scholze's construction crucially depends on the

insight that the (adic generic geometric fibre of the) Lubin-Tate tower "at the infinite level" can be seen as a perfectoid space, thereby pointing out the relevance of the last part of this thesis. However, we want to stress that, here, we will not go heavily into the theory of perfectoid spaces. Still, we will do all the computations necessary for someone acquainted with perfectoid spaces to hopefully conclude about the perfectoid nature of the Lubin-Tate tower.

This thesis

Finally, a few details about the approach of this thesis. This thesis only considers formal $\mathcal{O}_K$-modules, it does not consider p-divisible groups as is often done in the literature (when K has characteristic 0). As a result most arguments in this thesis are of a more computational nature.
Let us explain chapter by chapter what we do.
In Chapter 1 we define (1-dimensional) formal $\mathcal{O}_K$-modules and the basic objects one can define on them, such as: its Lie algebra, its module of invariant differentials, and (in some cases) its logarithm. We also construct "Lazard's ring", parametrizing all 1-dimensinal formal $\mathcal{O}_K$-module laws, and prove it is equal to a polynomial ring.
In Chapter 2, we compute the level 0 Lubin-Tate space. This will be done after developing some deformation theory. For example, we will prove a simple version of Schlessinger's deformation criterion from [Sch68]. It is only in this chapter that we introduce the concept of height. The final two sections, 2.5 and 2.6, are not necessary for the rest of this text, but serve to explore some constructions related to Lubin-Tate space. Specifically, Section 2.5 discussed "$\mathcal{O}_K$-typical coordinates" and Section 2.6 discusses (very roughly) the Gross-Hopkins period morphism.
In the last Chapter 3, the higher level Lubin-Tate spaces are defined. Then, we heavily rely on [Wei10] to show the perfectoid nature (in some sense) of the Lubin-Tate space "at infinite level". We should warn the reader: in our attempt to present the proof of [Wei10][Thm. 2.7.3] we did not understand a certain claim made by Weinstein. This leads to an unproven claim (3.5.20) which we feel forced to rely on to finish the "proof". As a consequence, the final proof of this thesis is not complete.
Acknowledgments: finally, before going on with mathematics, let me thank my advisor, Benjamin Schraen. He suggested this subject for my "memoire" and agreed to supervise it. Throughout this process, he has been available for help and he has shown great patience. For this, I want to thank him.
Conventions: before the reader starts, let us set a few conventions. As in the beginning of the introduction, K will always be "a non-Archimedean local field, with ring of integers $\mathcal{O}_K$, uniformizer π, and residue field k of characteristic p and cardinality q." This will be the case throughout this whole thesis. Most of our other conventions are conventions generally held by the mathematical community, such as what is meant by $\mathbb{Z}$ (integers) or $\mathbb{Z}_p$ (p-adic integers). We consider rings always to be commutative with 1. Whenever something is still ambiguous, we hope the context will clarify things.

Chapter 1

Formal $\mathcal{O}_K$-modules

1.1 Formal group laws

Formal group laws in nature

At first sight formal group laws might seem a bit artificial, but they actually occur quite naturally. Let us illustrate this by an example from differential geometry. If G is an n-dimensional Lie group (i.e. a smooth n-dimensional manifold with a smooth multiplication map $m\colon G \times G \to G$ making it into a group), then in local coordinates around the identity element e, the multiplication is a map

$$\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n .$$

We can Taylor expand this map around the origin (corresponding to e) to obtain the following description:

$$(\underline{X}, \underline{Y}) \coloneqq (X_1, \dots, X_n, Y_1, \dots, Y_n) \mapsto (F_1(\underline{X}, \underline{Y}), \dots, F_n(\underline{X}, \underline{Y})) =: F(\underline{X}, \underline{Y}),$$

where the $F_i(\underline{X}, \underline{Y})$ are power series in the variables $X_1, \dots, X_n, Y_1, \dots, Y_n$. Using the fact that G is a Lie group and that we Taylored around the identity, we find for every i that

$$F_i(\underline{X}, \underline{Y}) = X_i + Y_i + \text{higher order terms.} \tag{1.1}$$

Letting $\iota(\underline{X}) = (\iota_1(\underline{X}), \dots, \iota_n(\underline{X}))$ be the Taylor expansion of the inverse $G \to G, x \mapsto x^{-1}$, we obtain a power series satisfying

$$F(X, \iota(X)) = 0 = F(\iota(X), X). \tag{1.2}$$

Moreover, the associativity of G translates to $F(F(\underline{X}, \underline{Y}), \underline{Z}) = F(\underline{X}, F(\underline{Y}, \underline{Z}))$.

Viewing $F(\underline{X}, \underline{Y})$ as a formal power series, we have obtained an "n-dimensional formal group law over $\mathbb{R}$", which is to say, a formal power series with coefficients in $\mathbb{R}$ describing the multiplication of a group (locally around the identity).

Formal group laws in general

Abstracting the properties of the formal power series F we can give a general definition of formal group laws, over general rings R. As we are only interested

in commutative formal group laws in this thesis, we also add a commutativity condition[1].

Definition 1.1.1. *A* (commutative) n-dimensional formal group law *over* R *is an* n*-tuple* $F(\underline{X},\underline{Y}) = (F_1(\underline{X},\underline{Y}),\dots,F_n(\underline{X},\underline{Y}))$ *of formal power series in* $2n$*-variables* $F_i(\underline{X},\underline{Y}) \in R[\![\underline{X},\underline{Y}]\!] = R[\![X_1,\dots,X_n,Y_1,\dots,Y_n]\!]$ *satisfying the following conditions*

- $F(\underline{X},0) = \underline{X} = F(0,\underline{X})$ *(neutral element)*
- $F(\underline{X},F(\underline{Y},\underline{Z})) = F(F(\underline{X},\underline{Y}),\underline{Z})$ *(associativity)*
- $F(\underline{X},\underline{Y}) = F(\underline{Y},\underline{X})$ *(commutativity).*

From these three axioms it follows that the F_i have the form of 1.1 and that there is a unique $\iota(X) \in R[\![\underline{X}]\!]$ satisfying 1.2.
To simplify notation (and to stress the analogy with groups) we will often write

$$\underline{X} +_F \underline{Y},$$

instead of $F(\underline{X},\underline{Y})$. Similarly, we might write $-_F X = \iota(X)$ or $X -_F Y = X +_F \iota(Y)$.

Two important examples are the following.

Example 1.1.2. *The* (1-dimensional) additive formal group law $\mathbb{G}_a(X,Y)$ *over* R *is given by*

$$\mathbb{G}_a(X,Y) \coloneqq X + Y.$$

The n-dimensional additive formal group law $\mathbb{G}_a^n$ *over* R *is given by*

$$\mathbb{G}_a^n(\underline{X},\underline{Y}) \coloneqq \underline{X} + \underline{Y} = (X_1 + Y_1,\dots,X_n + Y_n).$$

Example 1.1.3. *The* multiplicative formal group law $\mathbb{G}_m(X,Y)$ *is given by*

$$\mathbb{G}_m(X,Y) \coloneqq X + Y + XY = (X+1)(Y+1) - 1.$$

Contrary to what one might naively expect, the formula is not simply XY*: remember that the multiplication should be centered around the identity element (if we Taylor the multiplication* $\mathbb{R}^\times \times \mathbb{R}^\times \to \mathbb{R}^\times$ *around* 1*, this formula is indeed what we get).*

Finally, we can define morphisms. They are exactly what you would expect, and what you would indeed obtain by Tayloring a morphism of Lie groups.

Definition 1.1.4. *Let* F *and* G *be formal group laws over* R *of dimensions* n *and* m *respectively. Then a* morphism $f\colon F \to G$ *from* F *to* G *is an* m*-tuple of formal power series in* n*-variables without constant terms* $f(\underline{X}) = (f_1(\underline{X}),\dots,f_m(\underline{X}))$ *transforming the addition defined by* F *into the addition defined by* G*:*

$$f(\underline{X} +_F \underline{Y}) = f(\underline{X}) +_G f(\underline{Y}).$$

The composition *of two morphisms* $f(\underline{X})$ *and* $g(\underline{Y})$ *is defined to be the (tuple of) power series* $g(f(\underline{X}))$ *and this way we obtain the* category of (commutative, finite dimensional) formal group laws over R.
An isomorphism *of formal group laws is then simply an isomorphism in this category (this is a morphism which is an invertible formal power series).*

[1] When R is reduced, every 1-dimensional formal group law over R is commutative [Laz55, Théorème I]. Being primarily interested in 1-dimensional ones, we are thus not being very restrictive.

Formal groups

Just as another choice of coordinates on the Lie group results in a different Taylor expansion of the multiplication, it is possible that two formal group laws "come from the same group" (i.e. they are isomorphic). The concept of *formal group* will provide us with a coordinate free perspective.

Let us start with the observation that if F is an n-dimensional formal group law over R and A a complete, *adic* R-algebra (i.e. a complete topological R-algebra whose topology is the I-adic one for some ideal I of A[2]), then the set of continuous morphisms of topological R-algebras

$$\mathrm{Spf}(R[\![\underline{X}]\!])(A) = \mathrm{Hom}_{\mathrm{ct}}(R[\![X_1, \ldots, X_n]\!], A),$$

where $R[\![\underline{X}]\!]$ has the $(\underline{X})$-adic topology, has a natural group structure given by F: a continuous morphism f is naturally identified with an n-tuple $(a_1, \ldots, a_n)$ of topologically nilpotent elements of A; the power series F tells us how to add two such n-tuples and this defines a group. We denote the group we obtain by $F(A)$ and this construction is functorial in A. We arrive at the following definition:

Definition 1.1.5. *An* n-dimensional (commutative) formal group *over* R *is a (commutative) cogroup object*[3] *in the category of complete adic* R*-algebras (the morphisms being continuous* R*-algebra morphisms) that is isomorphic to* $R[\![X_1, \ldots, X_n]\!]$.

A formal group over R determines a formal group law as soon as an isomorphism with $R[\![\underline{X}]\!]$ has been chosen. This is what we call "choosing coordinates".
Explicitly, when we know how $R[\![X_1, \ldots, X_n]\!]$ is a cogroup object we can consider (by the Yoneda lemma and the fact that the coproduct of $R[\![\underline{X}]\!]$ with itself exists in the category of complete adic R-algebras – being equal to $R[\![\underline{X}, \underline{Y}]\!]$) the induced comultiplication morphism

$$\mu\colon R[\![\underline{X}]\!] \to R[\![\underline{X}, \underline{Y}]\!],$$

which defines a formal group law by setting $F_i(\underline{X}, \underline{Y}) \coloneqq \mu(X_i)$.
Finally, note that a morphism of formal groups is a morphism of group functors/objects and that this is the same as a morphism of formal group laws after having chosen coordinates.

Remark 1.1.6. *Often, when we are a bit lazy or when we simply forget to be precise, we say "formal group" instead of "formal group law". Confusing one for another is only a problem when a construction depends on a particular choice of coordinates. The author hopes he will be sufficiently clear whenever such a situation arises.*

1.2 Invariant derivations and invariant differentials

We can consider the tangent space of a formal group, and as with Lie groups its elements can be interpreted as "invariant derivations". Dually, there is the concept

[2] Any such I is called an *ideal of definition*.
[3] That is, it(s formal spectrum) is a group object in the opposite category.

of "invariant differentials". Both are important invariants of formal group, and they basically encode the same information, by duality. For this reason it suffices to study one in detail and then using duality the reason about the other.

Classically, the *tangent space of* $R[\![X_1, \ldots, X_n]\!]$ *(over* R*)* is defined as the R-module of homomorphisms

$$\mathrm{Hom}_{\mathbf{Mod}_R}(P/P^2, R), \tag{1.3}$$

where $P = (\underline{X}) = (X_1, \ldots, X_n)$. We would like to think of this as the R-module

$$\mathrm{Hom}_{\mathrm{ct}, R-\mathbf{Alg}}(R[\![X_1, \ldots, X_n]\!], R[\varepsilon]),$$

where $R[\varepsilon]$ is defined by the relation $\varepsilon^2 = 0$ and has the discrete topology, but this is not always possible. In general, we have a canonical morphism

$$\mathrm{Hom}_{\mathbf{Mod}_R}(P/P^2, R) \to \mathrm{Hom}_{\mathrm{ct}, R-\mathbf{Alg}}(R[\![X_1, \ldots, X_n]\!], R[\varepsilon]),$$

which sends a morphism $f\colon P/P^2 \to R$ to the morphism $R[\![\underline{X}]\!] \to R[\varepsilon]$ that sends X_i to $f(X_i + P^2)\varepsilon$. This map is certainly injective, but it might not be surjective: indeed, when R has non-trivial nilpotents, we can get more morphisms $R[\![\underline{X}]\!] \to R[\varepsilon]$ than the ones obtained via this map. However, in the case that R is reduced, the map is easily seen to be an isomorphism.
So when F is a formal group and R is reduced, we can define its *tangent space* simply to be $F(R[\varepsilon])$[4], which is functorial in F.

Remark 1.2.1. *In the reduced case, note that* $F(R[\varepsilon])$ *has a group structure for two different reasons: by* F *being a group functor and because it is an* R*-module. Due to the fact that* ε *squares to* 0 *and* $F_i(\underline{X}, \underline{Y}) = X_i + Y_i +$ (*higher order terms*) *it follows that the group structures agree.*

In general, if R is not reduced, we can only identify the tangent space 1.3 with a subset of $F(R[\varepsilon])$, but this is a bit awkward and luckily there is another description in terms of invariant derivations.
Recall first that an *R-derivation* $D\colon R[\![\underline{X}]\!] \to R[\![\underline{X}]\!]$ is an R-linear map satisfying the Leibniz rule

$$D(fg) = fDg + gDf.$$

Since for all N one has $D(\underline{X})^N \subset (\underline{X})^{N-1}$ any R-derivation D is automatically continuous, and it follows that for every f

$$Df = \sum_i \frac{\partial f}{\partial X_i} DX_i,$$

so that the *$R[\![\underline{X}]\!]$-module of R-derivations of* $R[\![\underline{X}]\!]$, $\mathrm{Der}(R[\![\underline{X}]\!])$, equals

$$R[\![\underline{X}]\!]\frac{\partial}{\partial X_1} \oplus \cdots \oplus R[\![\underline{X}]\!]\frac{\partial}{\partial X_n}.$$

[4]As we will see in Section 2.1, this is similar to a more general notion of tangent spaces of certain types of functors.

A sub-R-module of it will be that of invariant derivations, but we choose to define those in terms of invariant differentials. So first recall the free $R[\![\underline{X}]\!]$-*module of* R-*differentials* $\Omega^1_{R[\![\underline{X}]\!]} \coloneqq \Omega^1_{R[\![\underline{X}]\!]/R}$, with basis $dX_1, \dots dX_n$. Let us define the R-linear continuous map

$$d\colon R[\![\underline{X}]\!] \to \Omega^1_{R[\![\underline{X}]\!]}, f \mapsto \sum_i \frac{\partial f}{\partial X_i} dX_i,$$

which satisfies the Leibniz rule.
Using this we can evaluate a differential $\omega = \sum_i f_i dX_i$ at an n-tuple of formal power series without constant terms $g(\underline{Y}) = (g_1(\underline{Y}), \dots, g_n(\underline{Y}))$ (in possibly another number of variables than n):

$$\omega(g)\colon = \sum_i f_i(g(\underline{Y}))dg_i \in \Omega^1_{R[\![\underline{Y}]\!]}.$$

Definition 1.2.2. *Let $F \in R[\![\underline{X}, \underline{Y}]\!]$ be an n-dimensional formal group law. An* invariant differential (for F) $\omega \in \Omega^1_{R[\![\underline{X}]\!]}$ *is a differential satisfying*

$$\omega(F(\underline{X}, \underline{Y})) = \omega(\underline{X}) + \omega(\underline{Y}).$$

The R-module of invariant differentials of F *is denoted by* $\omega(F)$.

Note that $\omega(-)$ is contravariant functor: if f is a morphism between two formal group laws F and G, and if $\omega \in \omega(G)$, then $\omega(f) \in \omega(F)$:

$$\omega\left(f(\underline{X} +_F \underline{Y})\right) = \omega\left(f(\underline{X}) +_G f(\underline{Y})\right) = \omega\left(f(\underline{X})\right) + \omega\left(f(\underline{Y})\right).$$

In this case, we might also write $f^*\omega$ for $\omega(f)$.

Example 1.2.3. *In the case $F(X, Y) = X + Y + \dots$ is a formal group law of dimension* 1*, one has for any differential $\omega = f(X)dX$ the formula*

$$\omega(F(X,Y)) = f\left(F(X,Y)\right) dF(X,Y) = f\left(F(X,Y)\right) \left(\frac{\partial F(X,Y)}{\partial X} dX + \frac{\partial F(X,Y)}{\partial Y} dY\right).$$

Therefore, ω is invariant if and only if

$$f\left(F(X,Y)\right) \frac{\partial F(X,Y)}{\partial X} = f(X),$$

plugging in $Y = \iota(X)$, the inverse of X, this equation becomes

$$f(X) = f(0) \frac{\partial F(T, \iota(X))}{\partial T}\Big|_{T=X}.$$

From the following calculation, using the chain rule,

$$\begin{aligned}\frac{\partial F(T, \iota(X))}{\partial T}\Big|_{T=X} \cdot \frac{\partial F(U,X)}{\partial U}\Big|_{U=0} &= \left(\frac{\partial F(T, \iota(X))}{\partial T}\Big|_{T=F(U,X)} \cdot \frac{\partial F(U,X)}{\partial U}\right)\Bigg|_{U=0} \\ &= \frac{\partial F\left(F(U,X), \iota(X)\right)}{\partial U}\Big|_{U=0} = \frac{\partial(U +_F X -_F X)}{\partial U}\Big|_{U=0} = 1\end{aligned}$$

it follows that if ω is invariant, then

$$f(X) = f(0) \left(\frac{\partial F(U,X)}{\partial U}\Big|_{U=0}\right)^{-1}.$$

Note that $\frac{\partial F(U,X)}{\partial U}\big|_{U=0} = 1 + (\textit{higher order terms})$ *(another reason it is invertible). Again by the chain rule, one can check that*

$$\omega_F \colon = \left(\frac{\partial F(U,X)}{\partial U}\Big|_{U=0}\right)^{-1} dX \in R[\![X]\!]dX$$

is indeed an invariant differential, from which it follows that

$$\omega(F) = R \cdot \omega_F.$$

Remark 1.2.4. *From now on we shall use the notation*

$$F_1(X,Y) = \frac{\partial F(T,Y)}{\partial T}\Big|_{T=X}$$

(and similarly use a subscript "2" in the case we are partially deriving in the second component). This should considerably simplify the above type of equations in the future.

The example turns out to have a generalization to the n-dimensional setting:

Proposition 1.2.5. *When F is an n-dimensional formal group law, $\omega(F)$ is a free R-module of rank n with a canonical basis $\omega_1, \ldots, \omega_n$ satisfying*

$$\omega_i \equiv dX_i \mod (\underline{X})\Omega^1_{R[\![\underline{X}]\!]}.$$

Proof. This is done by elementary computations, generalizing the ones above. A complete proof is given in [Zin84][Thm 1.19], an incomplete proof in [GH94][Prop. 2.2], and no proof is given here. ■

It follows that given a morphism of formal group laws $f \colon F \to G$, the induced morphism $f^* \colon \omega(G) \to \omega(F)$ is given by the matrix

$$\begin{pmatrix} \frac{\partial f_1}{\partial X_1}\Big|_{\underline{X}=0} & \frac{\partial f_2}{\partial X_1}\Big|_{\underline{X}=0} & \cdots \\ \frac{\partial f_1}{\partial X_2}\Big|_{\underline{X}=0} & \ddots & \\ \vdots & & \end{pmatrix} \tag{1.4}$$

with respect to the canonical bases.

Note that there is a canonical perfect pairing between the $R[\![\underline{X}]\!]$-modules of differentials and derivations:

$$\langle \cdot, \cdot \rangle \colon \Omega^1_{R[\![\underline{X}]\!]} \times \mathrm{Der}(R[\![\underline{X}]\!]) \to R[\![\underline{X}]\!], \left(\sum_i f_i dX_i, \sum_i g_i \frac{\partial}{\partial X_i}\right) \mapsto \sum_i f_i g_i, \tag{1.5}$$

which satisfies $\langle gdf, D\rangle = gDf$. It allows us to define the tangent space/invariant derivations of F in terms of $\omega(F)$:

Proposition 1.2.6 ([Zin84][Thm. 1.19])**.** *The R-module of R-derivations D for which*

$$\langle \omega(F), D\rangle \subset R$$

is free of rank n and called the R-module of invariant derivations *or simply* the Lie algebra of F.[5] *It is canonically isomorphic to the tangent space of F as defined in 1.3 and we denote it by* $\operatorname{Lie} F$.
The pairing 1.5 induces a perfect pairing of R-modules between $\omega(F)$ and $\operatorname{Lie} F$, *and if ω is any differential for which*

$$\langle \omega, \operatorname{Lie} F \rangle \subset R,$$

then it is invariant.

Proof. This follows from §1.2 and Thm. 1.19 of [Zin84]. The isomorphism with the tangent space follows from the other statements of this proposition and is functorial: both $\operatorname{Lie} F$ (as the dual of $\omega(F)$ with respect to the pairing) and $\operatorname{Hom}_{R\text{-Mod}}((\underline{X})/(\underline{X})^2, R)$ are n-dimensional; identifying them via their canonical bases gives us a canonical isomorphism of R-modules. If $f\colon F \to G$ is a morphism of formal group laws, then the induced morphism on the classical tangent space is given by the usual matrix, with respect to the canonical bases, which happens to be the transpose of the matrix 1.4. This is the same matrix as that of the induced morphism on the invariant differentials (because this is by definition the dual of the morphism $f^*\colon \omega(G) \to \omega(F)$), and therefore the isomorphisms are functorial. ■

Finally, let us note that we can speak of the Lie algebra and of the module of invariant differentials of a formal group, instead of of a formal group *law*, even though our constructions depended on a choice of coordinates. This is due to the fact that an isomorphism of formal group laws induces a canonical isomorphism on the Lie algebras/invariant differentials. If one is still not happy with this explanation, one could define, for a formal group represented by the complete adic ring A, the tangent space as the R-module of continuous R-linear morphisms $A \to \varepsilon R[\varepsilon]$ respecting multiplication (or indeed like in 1.3, using the *augmentation ideal* P of A).

1.3 Formal $\mathcal{O}_K$-module laws

Now let us study formal groups with additional structure. Recall that K is a non-Archimedean local field, with ring of integers $\mathcal{O}_K$, uniformizer π, and residue field $k = \mathbb{F}_q$ of cardinality q and characteristic p. Let R be an $\mathcal{O}_K$-algebra.

Definition 1.3.1. *An n-*dimensional formal $\mathcal{O}_K$-module law *F over R is an n-dimensional formal group law F over R together with a morphism of rings*

$$\mathcal{O}_K \to \operatorname{End}(F), a \mapsto a_F(X)$$

such that $a_F(X) = aX + $(higher order terms), or equivalenty, such that the induced map on the Lie algebra $\operatorname{Lie} F$ *(or on $\omega(F)$) is multiplication by a.*

Remark 1.3.2. *As before, we again have the concept of a* formal $\mathcal{O}_K$-module, *from which we obtain a formal $\mathcal{O}_K$-module law as soon as we choose coordinates. They can be defined as in 1.1.5 by replacing the word "group" by "$\mathcal{O}_K$-module" and, in addition, adding the condition on the Lie algebra. As before, we might, out of laziness or otherwise, talk about formal $\mathcal{O}_K$-modules when we actually mean formal $\mathcal{O}_K$-module laws.*

[5] We do not care about its bracket, which is trivial, because F is assumed commutative.

Example 1.3.3. *The* (1-dimensional) additive formal $\mathcal{O}_K$-module law $\mathbb{G}_a$ *is the additive formal group law* $\mathbb{G}_a$ *as before, with the* $a_{\mathbb{G}_a}(X)$ *given by* aX*. As before, there are also* n*-dimensional additive formal* $\mathcal{O}_K$*-module laws.*

Example 1.3.4. *When* $K = \mathbb{Q}_p$*, and thus* $\mathcal{O}_K = \mathbb{Z}_p$*, the multiplicative formal group law* $\mathbb{G}_m(X,Y) = (X+1)(Y+1)-1$ *over* $\mathbb{Z}_p$ *has a natural formal* $\mathbb{Z}_p$*-module structure. It necessarily has the following formal* $\mathbb{Z}$*-module structure (definition obvious)*

$$n_{\mathbb{G}_m}(X) = (X+1)^n - 1,\ n \geq 0,$$

because a formal $\mathbb{Z}$*-module is the same as a (commutative) formal group. Now, one can show that the assignment* $n \mapsto n_{\mathbb{G}_m}(X)$ *is continuous for the* p*-adic topology on* $\mathbb{Z}_{\geq 0}$ *and the* (X,p)*-adic topology on* $\mathrm{End}(\mathbb{G}_m) \subset \mathbb{Z}_p[\![X]\!]$*. Since being an endomorphism of* $\mathbb{G}_m$ *is a closed condition, it follows that* $\mathrm{End}(\mathbb{G}_m)$ *is complete for this topology, and since* $\mathbb{Z}_{\geq 0} \subset \mathbb{Z}_p$ *is dense, it follows that the map extends to* $\mathbb{Z}_p \to \mathrm{End}(\mathbb{G}_m)$*. By continuity the map is still additive, and by additivity, it has the required form on the Lie algebra.*
Note that the roots of the polynomials

$$p^n_{\mathbb{G}_m}(X) = (X+1)^{p^n} - 1$$

corresponding to powers of the uniformizer p *are the shifted* p^n*-th roots of unity* $\zeta_{p^n} - 1$*. The extension* K_∞ *obtained by adjoining all these roots to* $\mathbb{Q}_p$ *is totally ramified and by the (local) Kronecker-Weber theorem*

$$\mathbb{Q}_p^{ab} = \mathbb{Q}_p^{unr} \cdot K_\infty.$$

Indeed, the formal $\mathbb{Z}_p$*-module* $\mathbb{G}_m$ *is the one discussed in the introduction. It is an example of a "Lubin-Tate formal group" for* $\mathbb{Q}_p$*, used in the Lubin-Tate approach of local class field theory!*

The logarithm

Let us take a slight step back now, and consider again a formal group law $F(X,Y)$, of dimension 1 over a $\mathbb{Q}$-algebra R. As we have computed in Example 1.2.3 its invariant differentials $\omega(F)$ are generated by

$$\omega_F = \frac{1}{F_1(0,X)} dX = (1 + \cdots) dX.$$

Since the coefficients are in $R \supset \mathbb{Q}$, we can integrate this form to a unique power series with no constant term

$$f(X) = \int \omega \in XR[\![X]\!].$$

We claim that f is an isomorphism $F \xrightarrow{\sim} \mathbb{G}_a$. Indeed, by construction we have

$$d\left(f\left(F(X,Y)\right)\right) = \omega\left(F(X,Y)\right) = \omega(X) + \omega(Y) = d\left(f(X) + f(Y)\right).$$

Thus, $f\left(F(X,Y)\right) - f(X) - f(Y)$ is a formal power series both of which its partial derivatives vanish. Therefore it is a constant. Having no constant term it must be zero and we see that f is a morphism of formal groups. Since $f(X) = X + \cdots$ it is an invertible power series and hence an isomorphism of formal groups as claimed. The power series f is the unique isomorphism $F \xrightarrow{\sim} \mathbb{G}_a$ such that $f^*\omega_{\mathbb{G}_a} = df = \omega_F$ and is called the *logarithm of* F, also denoted by $\log_F$.

Example 1.3.5. *When $F = \mathbb{G}_a$, then $\log_{\mathbb{G}_a}(X) = X$.*

Example 1.3.6. *When $F = \mathbb{G}_m$, we have*

$$\frac{1}{F_1(0,X)} = \frac{1}{1+X} = \sum_{i\geq 0}(-1)^i X^i$$

and thus it follows that

$$\log_{\mathbb{G}_m}(X) = \sum_{i\geq 1}\frac{(-1)^{i+1}}{i}X^i,$$

the ordinary logarithm, expanded around 1*.*

Remark 1.3.7. *Note that the logarithm encodes our formal group law:*

$$F(X,Y) = f^{-1}\left(f(X) +_{\mathbb{G}_a} f(Y)\right) = f^{-1}\left(f(X) + f(Y)\right).$$

Therefore, given a power series of the form

$$g(X) = X + b_1X^2 + b_2X^3 + \cdots \in R[\![X]\!] \tag{1.6}$$

we can associate a formal group law to it by the formula

$$F_g(X,Y) = g^{-1}\left(g(X) + g(Y)\right).$$

By uniqueness, its logarithm will be equal to g. This gives a bijective correspondence between 1*-dimensional formal group laws and power series g as in 1.6. As a consequence the functor on $\mathbb{Q}$-algebras sending R to the set of* 1*-dimensional formal group laws over it is represented by the polynomial ring*

$$\mathbb{Q}[b_1, b_2, b_3, \dots],$$

a fact that we will see holds in greater generality.

For general formal $\mathcal{O}_K$-modules F (of arbitrary dimension) something similar happens, as long as the ring R does not have π-power torsion (i.e., if it is flat over $\mathcal{O}_K$). In this case we can identify R with a subring of $R \otimes_{\mathcal{O}_K} K$.

Proposition 1.3.8 ([GH94][Prop. 3.2])**.** *Let R be a flat $\mathcal{O}_K$-algebra and F a* 1*-dimensional formal $\mathcal{O}_K$-module law over R. Let $\omega \in \omega(F)$ be an invariant differential of F over R. Then there exists a unique morphism of formal $\mathcal{O}_K$-module laws over $R \otimes_{\mathcal{O}_K} K$*

$$f\colon F \to \mathbb{G}_a$$

such that $df = \omega$.

Proof. We may assume R is in fact a K-algebra. If the characteristic of K is 0, then ω has a unique primitive f without constant term. In our discussion of the logarithm over $\mathbb{Q}$ we already saw that f is a morphism of formal groups $F \to \mathbb{G}_a$. It is a morphism of formal $\mathcal{O}_K$-modules as well: indeed, $d\left(f\left(a_F(X)\right) - af(X)\right) = 0$ again, for all $a \in \mathcal{O}_K$.
The case where $K \supset k$ has characteristic $p > 0$ is a bit tricky as we use a result we will only prove later, so we advise the reader to come back later to read the

proof (and check it is not circular).
We know by the Corollary 2.3.9 that F is, after a change of coordinates, equal to $\mathbb{G}_a$ as a "formal k-module". In this case, we know that $\omega = rdX$ for some $r \in R$. Because of this, and because f should be a morphism of "formal k-modules", we see that f must have the form

$$rX + \sum_{i>0} a_i X^{q^i}.$$

We want that $\pi f = f \circ \pi_F$, i.e.

$$\pi \sum_{i>0} a_i (1 - \pi^{q^i - 1}) X^{q^i} = r\left(\pi_F(X) - \pi X\right) + \sum_{i>0} a_i \left(\pi_F(X) - \pi X\right)^{q^i},$$

which can be solved inductively, in a unique way (for this we note that $\pi_F(X)$ is also of the form $\sum_i b_i X^{q^i}$, being a morphism of "formal k-modules"). Since f respects the formal multiplication by π and is a morphism of "formal k-modules", it is a morphism of formal $k[\![\pi]\!] = \mathcal{O}_K$-modules as was to be shown. ■

If F is a formal $\mathcal{O}_K$-module law over R (still $\mathcal{O}_K$-flat) of dimension 1, the unique morphism $f\colon F \to \mathbb{G}_a$ over $R \otimes_{\mathcal{O}_K} K$ with $df = \omega_F = (1 + \cdots)dX$ is called the *logarithm of* F and denoted by $\log_F$. Because $\log_F(X) = X + \cdots$ we see that it must be an isomorphism. Therefore we get a generalization of Remark 1.3.7:

Corollary 1.3.9. *Every 1-dimensional formal $\mathcal{O}_K$-module over a K-algebra is isomorphic to $\mathbb{G}_a$. The polynomial K-algebra*

$$K[b_1, b_2, \dots]$$

represents the functor sending a K-algebra to the set of 1-dimensional formal $\mathcal{O}_K$-module laws over this K-algebra.

In the case R is π-adically complete (and still flat over $\mathcal{O}_K$), we have a nice formula for the logarithm:

Proposition 1.3.10. *Let R be a flat $\mathcal{O}_K$-algebra that is π-adically complete. Let F be a 1-dimensional formal group law over R. Then*

$$\log_F(X) = f(X) \coloneqq \lim_{n\to\infty} \frac{\pi_F^n(X)}{\pi^n},$$

where the limit is for the π-adic topology on each of the coefficients and is taken in $(R \otimes_{\mathcal{O}_K} K)[\![X]\!] = R[1/\pi][\![X]\!]$.[6]

Proof. If we can prove the limit to converge, we are done: indeed, one then has

$$\begin{aligned} f(X +_F Y) = \lim_{n\to\infty} \frac{\pi_F^n(X +_F Y)}{\pi^n} &= \lim_{n\to\infty} \frac{\pi_F^n(X) +_F \pi_F^n(Y)}{\pi^n} \\ &= \lim_{n\to\infty} \frac{\pi_F^n(X) + \pi_F^n(Y) + O\left((\pi_F^n(X), \pi_F^n(Y))^2\right)}{\pi^n} \\ &= \lim_{n\to\infty} \frac{\pi_F^n(X) + \pi_F^n(Y)}{\pi^n} \\ &= f(X) + f(Y), \end{aligned}$$

[6]The "π-adic topology" on $R[1/\pi]$ is the one where a basis of neighbourhoods of 0 is given by the $\pi^n R$ for $n \geq 0$. One can check that this makes $R[1/\pi]$ into a complete topological ring.

where $O\left((\pi_F^n(X), \pi_F^n(Y))^2\right)$ denotes an element in $(\pi_F^n(X), \pi_F^n(Y))^2 R[\![X]\!]$. When $\lim_{n\to\infty} \pi_F^n(X)/\pi^n$ exists, then $O\left((\pi_F^n(X), \pi_F^n(Y))^2\right)/\pi^n$ must tend to 0 as $n \to \infty$. This justifies the second last step in the above equations.
If the limit converges, we similarly get $f\left(a_F(X)\right) = a \cdot f(X)$ for all $a \in \mathcal{O}_K$; and clearly also $f(X) = X + \cdots$, so that $df = \omega_F$, and f must be the logarithm.
Therefore, we need to show the limit converges.
For this, first note that

$$\pi_F(X) \in (\pi X, X^2). \tag{1.7}$$

Substituting $\pi_F(X)$ for X in the above equation one obtains

$$\pi_F^2(X) \in (\pi^2 X, \pi X^2, X^4).$$

Making the same substitution again while remembering 1.7 we get

$$\pi_F^3(X) \in (\pi^3 X, \pi^2 X^2, \pi X^4, X^8)$$

and by induction we get for all n

$$\pi_F^n(X) \in (\pi^n X, \pi^{n-1} X^2, \ldots, \pi X^{2^{n-1}}, X^{2^n}) =: I_n.$$

Thus

$$\frac{\pi_F^n(X)}{\pi^n} \in \frac{1}{\pi^n} I_n = \left(X, \frac{1}{\pi} X^2, \frac{1}{\pi^2} X^4, \ldots, \frac{1}{\pi^n} X^{2^n}\right),$$

from which it follows that all the $\pi_F^n(X)/\pi^n$ are included in the (π, X)-adically complete $R[\![X]\!]$-module

$$M \coloneqq R \cdot X \times \frac{1}{\pi} R \cdot X^2 \times \frac{1}{\pi} R \cdot X^3 \times \frac{1}{\pi^2} R \cdot X^4 \times \cdots \subset (R \otimes_{\mathcal{O}_K} K)[\![X]\!].$$

Now notice that

$$\frac{\pi_F^{n+1}}{\pi^{n+1}} - \frac{\pi_F^n(X)}{\pi^n} = \frac{\pi_F(\pi_F^n(X)) - \pi \cdot \pi_F^n(X)}{\pi^{n+1}} \in \frac{1}{\pi} M \cdot I_n \subset \frac{1}{\pi} M \cdot (\pi, X)^n.$$

Since $\dfrac{1}{\pi} M$ is also (π, X)-adically complete, it follows that the limit converges. ■

1.4 Lazard's ring

Instead of Tayloring the multiplication map of a Lie group or elliptic curve to obtain formal group laws, one can also construct formal group laws algebraically, using the axioms at hand. Writing

$$F(X, Y) = X + Y + \sum_{i,j} a_{ij} X^i Y^j,$$

we easily see that the functor

$$\mathbf{Ring} \to \mathbf{Set}, R \mapsto \{\text{1-dimensional formal group laws over } R\}$$

is represented by

$$\Lambda \coloneqq \mathbb{Z}[a_{2,0}, a_{1,1}, a_{0,2}, a_{3,0}, \ldots]/(\text{relations}),$$

where the relations are those obtained from the axioms in Definition 1.1.1 of a formal group law.
This means that $F(X,Y)$ is a formal group law over Λ and that any morphism f from Λ to some ring R gives us a 1-dimensional formal group law f_*F (the *base-change* of F to R) given by

$$f_*F(X,Y) = X + Y + \sum_{i,j} f(a_{ij}) X^i Y^j.$$

Because of this, the formal group law F is called *universal*.
We see that constructing a 1-dimensional formal group law over R is the same as giving a ring morphism $\Lambda \to R$. In Remark 1.3.7 we have seen that $\Lambda \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}[b_1, b_2, \dots]$ and actually it turns out that Λ itself is already isomorphic to the polynomial ring $\mathbb{Z}[t_1, t_2, \dots]$. This was first proven in [Laz55][Théorème II] and for this reason Λ is sometimes called "Lazard's ring". Finding morphisms from this polynomial ring to R is very easy, and thus – knowing the isomorphism – there is a straightforward way to construct 1-dimensional formal group laws over R.

More generally, we are able to construct the $\mathcal{O}_K$-algebra $\Lambda_{\mathcal{O}_K}$ representing the functor

$$\mathcal{O}_K - \mathbf{Algebra} \to \mathbf{Set}, R \mapsto \{\text{1-dimensional formal } \mathcal{O}_K\text{-module laws over } R\}$$

by modding out the polynomial $\mathcal{O}_K$-algebra

$$\mathcal{O}_K[a_{ij}, \alpha_i | \alpha \in \mathcal{O}_K] \tag{1.8}$$

by the relations we get between the

$$F(X,Y) = X + Y + \sum_{i,j} a_{ij} X^i Y^j \text{ and } \alpha_F(X) = \alpha X + \sum_i \alpha_i X^i,\ \alpha \in \mathcal{O}_K \tag{1.9}$$

from the axioms of 1-dimensional formal $\mathcal{O}_K$-module laws.
Again, we know that

$$\Lambda_{\mathcal{O}_K} \otimes_{\mathcal{O}_K} K \cong K[b_1, b_2, \dots],$$

by Corollary 1.3.9. The aim of this section is to explain why it is a polynomial ring even before tensoring, generalizing Lazard's work. We mention that this was first done by Drinfeld [Dri74][Prop. 1.4]. However, the ring $\Lambda_{\mathcal{O}_K}$ will still be called "Lazard's ring" too.

The first observation is that $\Lambda_{\mathcal{O}_K}$ is naturally a graded $\mathcal{O}_K$-algebra. The grading is as follows: on the polynomial algebra 1.8 we set

$$\deg a_{ij} = i + j - 1 \text{ and } \deg \alpha_i = i - 1, \tag{1.10}$$

a priori it might not be obvious why this defines a grading after modding out. To see this, we broaden our perspective a bit, and consider the ring

$$\mathcal{O}_K[a_{ij}, \alpha_i | \alpha \in \mathcal{O}_K][\![X, Y]\!],$$

which contains the elements 1.9. If we set the degrees of X and Y to be equal to -1 we get a grading on this bigger ring and the elements 1.9 will be homogeneous of

degree -1, just like the variables. Therefore, these elements, as functions, respect the grading.
To obtain the relations by which we mod out, we should study the equations expressing associativity, commutativity, distributivity, etc.:

$$\begin{gathered} F\left(F(X,Y),Z\right) - F\left(X,F(Y,Z)\right) = 0, \\ F(X,Y) - F(Y,X) = 0, \\ \alpha_F\left(F(X,Y)\right) - F\left(\alpha_F(X),\alpha_F(Y)\right) = 0, \\ \vdots \end{gathered}$$

Giving Z degree -1 as well, one sees that the elements on the left hand sides are all homogeneous of degree -1. We need to mod out by their coefficients, but they necessarily are homogeneous as well. This means that in order to get $\Lambda_{\mathcal{O}_K}$ we mod out by a homogeneous ideal. Therefore 1.10 defines a grading on $\Lambda_{\mathcal{O}_K}$.

The second step is to construct a graded surjective morphism

$$\mathcal{O}_K[t_1, t_2, \dots] \twoheadrightarrow \Lambda_{\mathcal{O}_K}, \tag{1.11}$$

where t_i has degree i.
To conclude from this morphism, one draws the following commutative diagram, obtained by tensoring the surjection:

$$\begin{array}{ccc} \mathcal{O}_K[t_1, t_2, \dots] & \twoheadrightarrow & \Lambda_{\mathcal{O}_K} \\ \downarrow & & \downarrow \\ K[t_1, t_2, \dots] & \twoheadrightarrow & \Lambda_{\mathcal{O}_K} \otimes_{\mathcal{O}_K} K \end{array} . \tag{1.12}$$

We know that $\Lambda_{\mathcal{O}_K} \otimes_{\mathcal{O}_K} K \cong K[b_1, b_2, \dots]$. Let us make this isomorphism explicit: over K any (1-dimensional) formal $\mathcal{O}_K$-module G is determined by its logarithm via

$$G(X,Y) = \log_G^{-1}\left(\log_G(X) + \log_G(Y)\right) \text{ and } \alpha_G(X) = \log_G^{-1}\left(\alpha X\right).$$

The logarithm $\log_G$ is the image of the universal logarithm

$$X + b_1 X^2 + b_2 X^3 + \dots \in K[b_1, b_2, \dots][\![X]\!], \tag{1.13}$$

via the unique morphism $K[b_1, b_2, \dots] \to G$ defining G. Thus via the isomorphism $\Lambda_{\mathcal{O}_K} \otimes_{\mathcal{O}_K} K \cong K[b_1, b_2, \dots]$, we find that

$$\begin{gathered} F(X,Y) = f^{-1}\left(f(X) + f(Y)\right), \\ \alpha_F(X) = f^{-1}\left(\alpha X\right). \end{gathered} \tag{1.14}$$

This means that the isomorphism sends a universal coefficient a_{ij} or α_ℓ to whatever is the corresponding coefficient on the right hand aside in 1.14. If we again use our trick and give $K[b_1, b_2, \dots]$ a natural grading: we set the degree of X (and Y) to be -1 and give b_i degree i; then 1.13 becomes homogeneous of degree -1, and from this it follows that f (and thus also its inverse) respect grading, meaning that the right hand sides of 1.14 are graded of degree -1. Comparing coefficients on both sides of 1.14 shows that the a_{ij} and the α_ℓ are sent to homogeneous elements

of the same degree. To conclude: the isomorphism $\Lambda_{\mathcal{O}_K} \otimes_{\mathcal{O}_K} K \cong K[b_1, b_2, \dots]$ respects grading.
Finally, since the map 1.11 still respects grading after tensoring, we get a surjective graded morphism

$$K[t_1, t_2, \dots] \twoheadrightarrow \Lambda_{\mathcal{O}_K} \otimes_{\mathcal{O}_K} K \cong K[b_1, b_2, \dots].$$

Since the graded parts on both sides are of the same, finite dimension, we conclude it is an isomorphism, and in particular injective. From the commutativity of 1.12 it follows that the graded surjection 1.11 is actually an isomorphism too:

$$\mathcal{O}_K[t_1, t_2, \dots] \xrightarrow{\sim} \Lambda_{\mathcal{O}_K},$$

which is what we want(ed) to show.

We still need to construct said graded surjective morphism, and in order to do so, we need to analyze the (ordinary[7]) homogeneous parts of $F(X, Y)$ and the $\alpha_F(X)$ in more detail.
First, let us write

$$\Lambda_{\mathcal{O}_K} = \bigoplus_{i \geq 0} \Lambda_{\mathcal{O}_K, i},$$

for the decomposition into graded parts. Now set, for $n \geq 2$,

$$I_n \coloneqq (\Lambda_{\mathcal{O}_K,1} + \cdots + \Lambda_{\mathcal{O}_K,n-2})\, \Lambda_{\mathcal{O}_K}$$

(so $I_2 = 0$). This is the ideal generated by the coefficients of

$$\begin{aligned} F(X, Y) - X - Y \quad &\text{mod } (X, Y)^n, \\ \alpha_F(X) - \alpha X \quad &\text{mod } X^n. \end{aligned}$$

Let us now begin our analysis. We start with F: write

$$F(X, Y) = X + Y + \sum_n \tilde{C}_n(X, Y),$$

where $\tilde{C}_n(X, Y)$ is the homogeneous degree n polynomial part of F.
The homogeneous polynomial $\tilde{C}_n$ is clearly symmetric in both variables, and $F(X, 0) = X$ implies

$$\tilde{C}_n(X, 0) = 0 = \tilde{C}_n(0, Y). \tag{1.15}$$

The associativity condition together with the fact that $F(X, Y) = X + Y + \cdots$ shows that, after modding out by the preceding non-unit coefficients I_n, one has

$$\tilde{C}_n(X + Y, Z) = \tilde{C}_n(X, Y + Z). \tag{1.16}$$

One can try to solve these equations, but it is not entirely trivial. The solution, which is due to Lazard, turns out to be very nice:

[7] In the sense of polynomials.

Proposition 1.4.1 ([Laz55][Lemme 3])**.** *Any symmetric homogeneous degree n-polynomial over any ring R satisfying 1.15 and 1.16 is a multiple of the homogeneous polynomial with the same properties*

$$C_n(X,Y) \coloneqq \begin{cases} (X+Y)^n - X^n - Y^n, & \text{if } n \text{ is not a prime power} \\ \dfrac{(X+Y)^n - X^n - Y^n}{p'}, & \text{if } n \text{ is a power of the prime } p' \end{cases}$$

defined over $\mathbb{Z}$.

Proof. It is not hard to see that the C_n as defined satisfy the properties 1.15 and 1.16. The harder part is to show $\tilde{C}_n$ is a multiple of C_n, for this one should first write down a system of equations for the coefficients of $\tilde{C}_n$: do this by comparing the coefficients on both sides of 1.16.
The solution to this system is given in [Ans21][Lemma 2.20], where it is called "Lucas' Theorem". The proof is very nice and exactly what finishes the proof of this proposition; [Ans21] attributes it to Lurie('s notes [Lur10][Lecture 3]).
Although we just need this result for $\mathcal{O}_K$-algebras, it holds in general, and in fact the proof referenced in some sense makes a reduction to our case (the reduction is to $\mathbb{Z}_p$- or to $\mathbb{Q}$-algebras). ■

Something similar can be done with respect to the homogeneous degree n parts (i.e., degree n monomials) $\alpha_n X^n$ of the $\alpha_F(X)$, but now we need to consider them all at once and relate them to $\tilde{C}_n$ as well. For this, we follow [Dri74][§1]. From the fact that F is a formal $\mathcal{O}_K$-module, we find for all $\alpha, \beta \in \mathcal{O}_K$

$$\begin{gathered} \alpha\left(X + Y + \tilde{C}_n(X,Y)\right) + \alpha_n(X+Y)^n \equiv \\ \alpha X + \alpha Y + \alpha_n X^n + \alpha_n Y^n + \tilde{C}_n(\alpha X, \alpha Y), \end{gathered} \tag{1.17}$$

$$\begin{gathered} (\alpha+\beta)X + (\alpha+\beta)_n X^n \equiv \\ \alpha X + \beta X + \tilde{C}_n(\alpha X, \beta X) + \alpha_n X^n + \beta_n X^n, \end{gathered} \tag{1.18}$$

$$(\alpha\beta)X + (\alpha\beta)_n X^n \equiv \alpha\left(\beta X + \beta_n X^n\right) + \alpha_n \beta^n X^n, \tag{1.19}$$

where the congruences are all modulo I_n. (In what follows, all congruences will be modulo I_n.)
By Theorem 1.4.1 there is a γ_n (wlog in $\Lambda_{\mathcal{O}_K, n-1}$) such that

$$\tilde{C}_n \equiv \gamma_n C_n.$$

Thus, we obtain from 1.17 that

$$\alpha_n\left((X+Y)^n - X^n - Y^n\right) \equiv (\alpha^n - \alpha)\gamma_n C_n(X,Y).$$

Therefore, it follows from the fact that the coefficients of C_n are coprime (*but not pairwise*), that

$$(\alpha^n - \alpha)\gamma_n \equiv \begin{cases} \alpha_n, & \text{if } n \text{ is not a prime power} \\ \alpha_n p', & \text{if } n \text{ is a power of the prime } p' \end{cases}. \tag{1.20}$$

The second equation, 1.18, tells us that

$$(\alpha+\beta)_n - \alpha_n - \beta_n \equiv \gamma_n C_n(\alpha, \beta). \tag{1.21}$$

Finally, from the third equation, we obtain

$$(\alpha\beta)_n \equiv \alpha\beta_n + \alpha_n\beta^n, \tag{1.22}$$

and thus by symmetry

$$\alpha\beta_n + \alpha_n\beta^n \equiv \beta\alpha_n + \beta_n\alpha^n, \tag{1.23}$$

i.e.

$$(\alpha^n - \alpha)\beta_n \equiv (\beta^n - \beta)\alpha_n. \tag{1.24}$$

Now let us consider some cases:

- n is not a p-power: in this case the α_n are clearly congruent to multiples of γ_n, because of 1.20 and the fact that all primes p' different from p are invertible in $\mathcal{O}_K$.

- n is a p-power, but no q-power: then there exists an $\alpha \in \mathcal{O}_K$ such that

  $$\alpha^n - \alpha \in \mathcal{O}_K^\times.$$

  It follows from 1.24 that the β_n are congruent to multiples of α_n and the same holds for γ_n by 1.20.

- n is a q-power: let us write $\overline{x}$ for $x \mod \pi_n\mathcal{O}_K + I_n$. The equations 1.22 and 1.23 with $\beta = \pi$ become

  $$\overline{(\alpha\pi)}_n \overset{1.22}{=} \pi^n\overline{\alpha}_n \overset{1.23}{=} \pi\overline{\alpha}_n,$$

  so

  $$0 = (\pi^{n-1} - 1)\pi\overline{\alpha}_n = \underbrace{(\pi^{n-1} - 1)}_{\text{unit}}\overline{(\pi\alpha)}_n,$$

  whence $\overline{(\pi\alpha)}_n = 0$ for every $\alpha \in \mathcal{O}_K$. In particular,

  $$\begin{aligned} 0 = \overline{p}_n = \overline{(1 + (p-1))}_n &\overset{1.21}{=} \overline{1}_n + \overline{(p-1)}_n + C_n(1, p-1)\overline{\gamma}_n \\ &\overset{1.21}{=} 2 \cdot \overline{1}_n + \overline{(p-2)}_n + (C_n(1, p-2) + C_n(1, p-2))\,\overline{\gamma}_n \\ &\overset{1.21}{=} \cdots \\ &\overset{1.21}{=} p \cdot \overline{1}_n + \overline{0}_n + \sum_{i=0}^{p-1} C_n(1, i)\overline{\gamma}_n \\ &= \sum_{i=0}^{p-1} C_n(1, i)\overline{\gamma}_n, \end{aligned}$$

  where the last equation holds because $p \cdot \overline{1}_n = \overline{p}_n = 0 = \overline{0}_n$.[8] Now, over $\mathbb{Z}$,

  $$\sum_{i=0}^{p-1} C_n(1, i) = \sum_{i=0}^{p-1} \frac{(i+1)^n - i^n - 1}{p} = \frac{p^n - p}{p} = p^{n-1} - 1,$$

[8] Or simply because $1_F(X) = X$ and thus $1_n = 0$, likewise $0_n = 0$.

from which it follows that $\overline{\gamma}_n$ is 0.
Therefore,

$$M \coloneqq \Lambda_{\mathcal{O}_K,n-1}/\left((I_n + \pi_n\mathcal{O}_K) \cap \Lambda_{\mathcal{O}_K,n-1}\right)$$

is generated by the α_n for $\alpha \in \mathcal{O}_K$ and thus a k-module, since $\pi\overline{\alpha}_n = \overline{(\alpha\pi)}_n = 0$ for all α.
Now, since $\overline{\gamma}_n = 0$, 1.21 tells us that $h\colon \alpha \mapsto \overline{\alpha}_n$ is additive. Moreover, by 1.22 and the fact that $q|n$ this application satisfies the Leibniz rule. Since $h(\pi\mathcal{O}_K) = 0$, we see that h is simply "differentiation"

$$k \to M,$$

and therefore 0.
Thus $M = \operatorname{Im} h = 0$ and we see that the α_n and γ_n are all congruent to multiples of π_n.

The above three cases tell us that for $n \geq 2$ the

$$\Lambda_{\mathcal{O}_K,n-1}/\left(I_n \cap \Lambda_{\mathcal{O}_K,n-1}\right) \tag{1.25}$$

are generated over $\mathcal{O}_K$ by a single element. To finally obtain our graded surjection

$$\mathcal{O}_K[t_1, t_2, \dots] \twoheadrightarrow \Lambda_{\mathcal{O}_K},$$

with $\deg t_i = i$, we simply send t_i to any lift of a generator of $\Lambda_{\mathcal{O}_K,i}/\left(I_{i+1} \cap \Lambda_{\mathcal{O}_K,i}\right)$. This is certainly graded, and it is also surjective: we can check this by induction on the degree: since $I_2 = 0$ the map surjects on $\Lambda_{\mathcal{O}_K,1}$; since the map surjects on $\Lambda_{\mathcal{O}_K,2}$ modulo $I_3 = \Lambda_{\mathcal{O}_K,1}$, the map surjects on $\Lambda_{\mathcal{O}_K,2}$ even before modding out, etc.
We obtain:

Theorem 1.4.2 ([Laz55][Théorème II], [Dri74][Prop. 1.4])**.** $\Lambda_{\mathcal{O}_K} \cong \mathcal{O}_K[t_1, t_2, \dots]$.

The fact that we have this isomorphism implies that the 1.25 are torsion-free and thus isomorphic to $\mathcal{O}_K$. More precisely, taking our previous calculations into account:

Proposition 1.4.3 ([Dri74][Prop. 1.3])**.** *The $\mathcal{O}_K$-module*

$$\Lambda_{\mathcal{O}_K,n-1}/\left(I_n \cap \Lambda_{\mathcal{O}_K,n-1}\right)$$

is free of rank 1 *and can be generated by (the class of)*

$$u = \begin{cases} d_n\gamma_n, & \text{if } n \text{ is not a power of } p \\ \alpha_n/(\alpha^n - \alpha), & \text{if } n \text{ is a } p\text{-power, but not a } q\text{-power and } \alpha^n \neq \alpha \mod \pi \\ \pi_n/(\pi^{n-1} - 1), & \text{if } n \text{ is a } q\text{-power} \end{cases},$$

here $d_n = \begin{cases} 1, & \text{if } n \text{ is no prime power} \\ p', & \text{if } n \text{ is a power of the prime } p' \end{cases}$.
Moreover, with this choice of generators we have: if n is no q-power, then

$$\beta_n \equiv (\beta^n - \beta)u,\ \gamma_n C_n(X, Y) \equiv u\left((X+Y)^n - X^n - Y^n\right)$$

for all $\beta \in \mathcal{O}_K$; and if n is a q-power, then

$$\beta_n \equiv \frac{\beta^n - \beta}{\pi}u,\ \gamma_n \equiv \frac{p}{\pi}u$$

for all $\beta \in \mathcal{O}_K$.

Remark 1.4.4. *The isomorphism 1.4.2 is very nice to have, but unfortunately it is not canonical: for example, it already depends on the lifts we choose of the u of the above proposition!*
We can get a bit of control over it, by requiring that, whenever n is a q-power t_{n-1} is corresponds to π_n.[9] *This is a property we will in the future assume this map has.*

Corollary 1.4.5 ([Dri74][Prop. 1.5])**.** *Let $n \geq 1$ and let R be a $\mathcal{O}_K$-algebra and F and G two 1-dimensional formal $\mathcal{O}_K$-module laws over R such that*

$$F \equiv G,\ a_F \equiv a_G \mod \text{degree } n$$

for all $a \in \mathcal{O}_K$.
Then

- *<u>if n is no power of q:</u> there is an $r \in R$ such that for all $a \in \mathcal{O}_K$*

$$F(X,Y) \equiv G(X,Y) + r\left((X+Y)^n - X^n - Y^n\right) \mod \text{degree } n+1,$$
$$a_F(X) \equiv a_G(X) + r(a^n - a)X^n \mod \text{degree } n+1.$$

- *<u>if n is a power of q:</u> there is an $r \in R$ such that for all $a \in \mathcal{O}_K$*

$$F(X,Y) \equiv G(X,Y) + r\frac{p}{\pi}C_n(X,Y) \mod \text{degree } n+1,$$
$$a_F(X) \equiv a_G(X) + r\frac{a^n - a}{\pi}X^n \mod \text{degree } n+1.$$

[9] We could also put more requirements on the isomorphism, but we will not need this.

Chapter 2

Deformations and extensions

The theorem, due to Lubin and Tate for $\mathcal{O}_K = \mathbb{Z}_p$ ([LT66][Thm. 3.1]) and due to Drinfeld in general ([Dri74][Prop. 4.2]), that

$$\Lambda_{\mathcal{O}_K} \cong \mathcal{O}_K[t_1, t_2, \dots]$$

is very important for the deformation theory of 1-dimensional formal $\mathcal{O}_K$-modules. Namely, it says that the deformation theory is "unobstructed". Concretely, this means that if R is a $\mathcal{O}_K$-algebra with an ideal I and if F is a 1-dimensional formal $\mathcal{O}_K$-module over R/I, then there is no obstruction to being able to lift F to R: indeed, any morphism $\mathcal{O}_K[t_1, t_2, \dots] \to R/I$ lifts to a morphism to R, and this defines a lift (i.e. a *deformation*) $\tilde{F}$ of F over R.
What is most important to us is that the functor represented by $\Lambda_{\mathcal{O}_K}$ is "formally smooth", this means the following:

Definition 2.0.1. *A set-valued (covariant) functor F on a category of rings (containing all quotients by ideals of its objects) is called* formally smooth *if for every object R and every nilpotent ideal $I \subset R$ the induced morphism*

$$F(R) \twoheadrightarrow F(R/I)$$

is surjective.
By dévissage the condition that I be nilpotent can be replaced by the condition that $I^2 = 0$.

Example 2.0.2. *The functor represented by a formal group on the category of complete adic R-algebras as in 1.1.5 is formally smooth. Indeed, it is representable by $R[\![X_1, \dots, X_n]\!]$.*
Similarly, formal $\mathcal{O}_K$-modules are formally smooth.

The primary goal of this chapter is to compute the Lubin-Tate space at level 0. This is an affine formal scheme representing the functor that sends a certain kind of $\mathcal{O}_K$-algebra R to the set of deformation classes (deformations up to some kind of equivalence) of F over R, where F is a 1-dimensional formal $\mathcal{O}_K$-module law defined over k, of height h. We will define all these concepts later, just keep in mind that the points of Lubin-Tate space should be lifts of a certain formal $\mathcal{O}_K$-module law defined over k. Our aim is to show the Lubin-Tate space (at level 0) equals

$$\operatorname{Spf} \mathcal{O}_K[\![u_1, \dots u_{h-1}]\!]. \tag{2.1}$$

Because the deformation theory of 1-dimensional formal group laws is unobstructed, it will be easily seen that the Lubin-Tate deformation functor is formally smooth. Then the essential thing is to compute its "tangent space": if we can prove it is $h-1$-dimensional, there is a theorem due to Schlessinger [Sch68] that gives us the description 2.1. In the next section we will discuss Schlessinger's deformation theorem.

2.1 Schlessinger's deformation theorem

In this section we basically follow [Far05][§2], although we make some minor adjustments, because our setting is more general. We will prove, as [Far05] calls it, Schlessinger's deformation theorem "in the unobstructed case". The general version is indeed due to Schlessinger [Sch68][Thm. 2.11].

The type of $\mathcal{O}_K$-algebras we are interested in are *complete local Noetherian $\mathcal{O}_K$-algebras*, that is, local Noetherian $\mathcal{O}_K$-algebras R with maximal ideal $\mathfrak{m}_R$ containing (the image of) π that are complete for the $\mathfrak{m}_R$-adic topology.
Throughout this section, let R be a complete local Noetherian $\mathcal{O}_K$-algebra with maximal ideal $\mathfrak{m}_R$ and residue field k_R. We let

$$\mathcal{C}_R$$

be the category of local Artinian R-algebras with residue field k_R[1], with morphisms being local R-algebra morphisms. This category is very suitable for dévissage, because every non-unit ideal of an object A is nilpotent.

Remark 2.1.1. *Note that any $A \in \mathcal{C}_R$ is an R-module of finite length: since A is Artinian it is certainly of finite length over itself, so there is a maximal chain of ideals*

$$0 = I_0 \subsetneq I_1 \subsetneq \cdots \subsetneq I_n = A$$

such that the successive quotients

$$I_{i+1}/I_i$$

are all simple A-modules, i.e., isomorphic to k_R. Since k_R is a length 1 R-module, and since length is additive, it follows that A is of length n over R, being an extension of n length 1 R-modules.

It follows from the remark that fiber products exist in $\mathcal{C}_R$: suppose we have a diagram

$$\begin{array}{ccc} A & & B \\ & \searrow \quad \swarrow & \\ & C & \end{array} , \tag{2.2}$$

then the ordinary fiber product of rings $A \times_C B \subset A \times B$ belongs to the category $\mathcal{C}_R$. Indeed, it is local with maximal ideal the inverse image of the maximal ideal of C, and since A and B are of finite height over R, so is $A \times B$ and therefore so is $A \times_C B$: thus it is also Artinian. Finally, its residue field is clearly also k_R.
Therefore, the following definition makes sense:

[1] We require that the image of $\mathfrak{m}_R$ is contained in the maximal ideal of such a ring.

Definition 2.1.2. *A functor $F\colon \mathcal{C}_R \to \mathbf{Set}$ is said to satisfy the* Mayer-Vietoris property *if for all diagrams 2.2 the natural morphism*

$$F(A \times_B C) \to F(A) \times_{F(C)} F(B)$$

is an isomorphism.

Now, let $\mathbf{Vect}_{k_R}^{\mathrm{fd}}$ be the category of finite dimensional k_R-vector spaces. We define the following functor

$$\mathbf{Vect}_{k_R}^{\mathrm{fd}} \to \mathcal{C}_R, V \mapsto k_R \oplus V, \tag{2.3}$$

where the R-module $k_R \oplus V$ has an R-algebra structure by setting

$$(x, v) \cdot (y, w) = (xy, xw + yv).$$

This makes $k_R \oplus V$ an element of $\mathcal{C}_R$.
Let F now be any functor

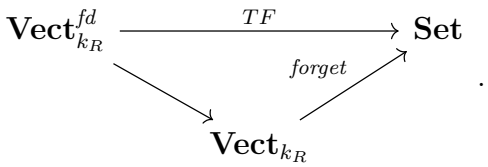
$$\mathcal{C}_R \to \mathbf{Set}.$$

We denote by TF the *tangent functor associated to* F, which is F composed after the functor above

$$TF\colon \mathbf{Vect}_{k_R}^{\mathrm{fd}} \to \mathbf{Set}, V \mapsto F(k_R \oplus V).$$

The image $TF(k_R) \cong F(k_R[\varepsilon])$ is called the *tangent space of* F. Notice that both the tangent functor and the tangent space are functorial in F.
When F satisfies the Mayer-Vietoris property, the tangent space itself encodes the whole tangent functor.

Proposition 2.1.3 ([Sch68][Lemma 2.10])**.** *Let $F\colon \mathcal{C}_R \to \mathbf{Set}$ satisfy the Mayer-Vietoris property. Then the functor TF is actually a functor to the category $\mathbf{Vect}_{k_R}$ of k_R-vector spaces, that is, there is a canonical factorization*



Moreover, there are natural isomorphisms

$$TF(V) \cong TF(k_R) \otimes_{k_R} V.$$

Proof. Every finite dimensional k_R-vector space V is a k_R-vector space. Therefore there is a morphism $V \times V \to V$ expressing addition and for every scalar $x \in k_R$ there is the scalar multiplication morphism $m_x\colon V \to V$. There are the obvious commutative diagrams expressing the axioms of k_R-vector spaces. Since the functor 2.3 respects products and so does F, it follows that these diagrams still commute for $TF(V)$. This naturally makes $TF(V)$ into a k_R-vector space and proves the claimed factorization.
For the second claim, pick an isomorphism $V \cong k_R^n = k_R \times \cdots \times k_R$, then

$$TF(V) \cong TF(k_R^n) \cong TF(k_R) \times \cdots \times TF(k_R) \cong TF(k_R) \otimes_{k_R} k_R^n \cong TF(k_R) \otimes_{k_R} V$$

and one can check that this isomorphism is natural in V. ∎

We will now introduce "formally étale" morphisms between set-valued functors on $\mathcal{C}_R$. It generalizes the notion one might already know for schemes.

Definition 2.1.4. *Let $F, G\colon \mathcal{C}_R \to \mathbf{Set}$ be covariant functors. A morphism $f\colon F \to G$ is called* formally étale *if for all ideals $I \subset A$, the following diagram is Cartesian*

$$\begin{array}{ccc} F(A) & \longrightarrow & G(A) \\ \downarrow & & \downarrow \\ F(A/I) & \longrightarrow & G(A/I) \end{array} .$$

As in the definition of formal smoothness, we can equivalently, by dévissage, only consider ideals I that square to 0 or even: that are killed by the maximal ideal of A.[2]

In a Cartesian square, the upper horizontal arrow is a bijection when the lower horizontal arrow is. Therefore we obtain the following lemma, by dévissage:

Lemma 2.1.5. *Let $F, G\colon \mathcal{C}_R \to \mathbf{Set}$ be covariant functors. A morphism $f\colon F \to G$ is an isomorphism if and only if it is formally étale and induces a bijection on the images of the final object:*

$$F(k_R) \xrightarrow{\sim} G(k_R).$$

Corollary 2.1.6. *A formally étale morphism $f\colon F \to G$ between functors both satisfying the Mayer-Vietoris property is an isomorphism.*

Proof. By Proposition 2.1.3, if F satisfies the Mayer-Vietoris property, then $F(k) = TF(0) = 0$, a singleton. Therefore, formally étale morphisms between functors both satisfying the Mayer-Vietoris property are isomorphisms. ∎

Proposition 2.1.7 ([Far05][1st Course, Prop. 7])**.** *Let $F, G\colon \mathcal{C}_R \to \mathbf{Set}$ be covariant functors both satisfying the Mayer-Vietor property and let $f\colon F \to G$ be a morphism. Suppose F is formally smooth. Then f is an isomorphism if and only if it induces an isomorphism on the tangent spaces.*

Proof. By Corollary 2.1.6 it is enough to show that if f induces an isomorphism on the tangent spaces, then f is formally étale.
By Proposition 2.1.3 we know f induces an isomorphism of tangent functors.
Now let A be an object of $\mathcal{C}_R$ and $I \subset A$ an ideal killed by the maximal ideal $\mathfrak{m}_A$ of A. By dévissage, it is enough to prove the diagram

$$\begin{array}{ccc} F(A) & \longrightarrow & G(A) \\ \downarrow\!\!\downarrow & & \downarrow \\ F(A/I) & \longrightarrow & G(A/I) \end{array}$$

is Cartesian. Because the left vertical arrow is surjective (F is formally smooth), it suffices to check that $F(A) \to G(A)$ induces isomorphisms between the fibers of the vertical arrows. These fibers are either both empty, or, by the next lemma, they are principal homogeneous spaces[3] under $TF(I) \cong TG(I)$ in a compatible way. In both cases, the fibers are isomorphic. ∎

[2] On the category $\mathcal{C}_R$ this condition can also be a replacement in the definition of formal smoothness.

[3] That is, they have a group acting on them, and this action is free and transitive.

Lemma 2.1.8 ([Far05][1st Course, Prop. 6]). *Let A be an object in $\mathcal{C}_R$ and $I \subset A$ and ideal killed by the maximal ideal $\mathfrak{m}_A$ of A. Let $F\colon \mathcal{C}_R \to \mathbf{Set}$ satisfy the Mayer-Vietoris property and let $f\colon F(A) \to F(A/I)$ be the morphism induced by the reduction modulo I.*
Then for every $x \in F(A/I)$ the fiber $f^{-1}(x)$ is either empty or a principal homogeneous space under $TF(I)$.

Proof. We have an isomorphism in $\mathcal{C}_R$

$$A \times_{A/I} A \xrightarrow{\sim} A \times_{k_R} (k_R \oplus I)$$

given by $(a_1, a_2) \mapsto (a_1, \overline{a}_1 \oplus (a_1 - a_2))$, where $\overline{a}_1 = (a_1 \mod \mathfrak{m}_A)$. The inverse is given by $(a, \overline{a} \oplus i) \mapsto (a, a - i)$. The projection on the second coordinate

$$\mathrm{pr}_2\colon A \times_{k_R} (k_R \oplus I) \xrightarrow{\sim} A \times_{A/I} A \to A$$

is therefore $(a, i) \mapsto a - i$. It trivially follows that the following diagram commutes:

$$\begin{array}{ccc} A \times_{k_R} (k_R \oplus I) \times_{k_R} (k_R \oplus I) & \xrightarrow{\mathrm{id} \times +} & A \times_{k_R} (k_R \oplus I) \\ \downarrow{\scriptstyle \mathrm{pr}_2 \times \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{pr}_2} \\ A \times_{k_R} (k_R \oplus I) & \xrightarrow{\mathrm{pr}_2} & A \end{array}, \tag{2.4}$$

where $+\colon (k_R \oplus I) \times_{k_R} (k_R \oplus I) \to (k_R \oplus I), (x \oplus i, x \oplus j) \mapsto (x \oplus i + j)$. By the Mayer-Vietoris property we obtain an isomorphism

$$F(A) \times TF(I) \cong F(A) \times_{F(A/I)} F(A), \tag{2.5}$$

which is moreover natural in F. Applying F to the diagram 2.4, we obtain diagram expressing a group action by $TF(I)$ on $F(A)$ by using the projection on the second coordinate

$$\mathrm{pr}_2\colon F(A) \times TF(I) \xrightarrow{\sim} F(A) \times_{F(A/I)} F(A) \to F(A).$$

By construction, the group action respects the fibers of f. From the isomorphism 2.5 we see that the action by $TF(I)$ is both free and transitive. ∎

We are now ready to prove Schlessinger's deformation theorem (in the unobstructed case).

Theorem 2.1.9 ([Sch68][Thm. 2.11], [Far05][Thm. 3]). *Let $F\colon \mathcal{C}_R \to \mathbf{Set}$ be a formally smooth functor satisfying the Mayer-Vietoris property and suppose*

$$\dim_{k_R} TF(k) = n < \infty.$$

Then F is prorepresented by $R[\![T_1, \ldots, T_n]\!]$, i.e. $F \cong \mathrm{Spf}\, R[\![T_1, \ldots, T_n]\!]$.

Proof. Let $(e_1, \ldots, e_n)$ be a basis for the tangent space. By the Mayer-Vietoris property this is an element η_1 of

$$F\left(k_R[T_1, \ldots, T_n]/(T_1, \ldots, T_n)^2\right) \cong \prod_{i=1}^{n} TF(k).$$

Since F is formally smooth, this element can be lifted to an element of

$$(\eta_1, \eta_2, \dots) \in F\left(R[\![T_1, \dots, T_n]\!]\right) \coloneqq \varprojlim_{\ell} F\big(\underbrace{R[\![T_1, \dots, T_n]\!]/I^{\ell}}_{\in \mathcal{C}_R}\big),$$

where $I = \mathfrak{m}_R + (T_1, \dots, T_n)^2$.
We obtain a morphism $f\colon G \coloneqq \operatorname{Spf} R[\![T_1, \dots, T_n]\!] \to F$: namely, an element $a \in G(A)$ corresponds to a morphism

$$\tilde{a}\colon R[\![T_1, \dots, T_n]\!]/I^{\ell} \to A$$

for ℓ big enough. Our morphism f sends the element a to $F(\tilde{a})(\eta_\ell)$. One checks that it does not depend on ℓ and that this indeed defines a natural transformation. Since G is formally smooth and satisfies the Mayer-Vietoris property, it suffices to prove that f induces an isomorphism on the tangent spaces, by Proposition 2.1.7. It induces an isomorphism on tangent spaces by construction: indeed, a basis for $TG(k_R) = \operatorname{Hom}(k_R[T_1, \dots, T_n]/(T_1, \dots, T_n)^2, k_R[\varepsilon])$ is given by the projections on the i-th coordinates

$$\mathrm{pr}_i\colon k_R[T_1, \dots, T_n]/(T_1, \dots, T_n)^2 \cong k_R[\varepsilon] \times_{k_R} \times \cdots \times k_R[\varepsilon] \to k_R[\varepsilon].$$

Since $F(\mathrm{pr}_i)(\eta_1) = e_i$, we are done. ■

Because of this theorem, it will be of great importance to compute the tangent space of the Lubin-Tate deformation functor.

2.2 First order deformations and extensions

To compute the "tangent space of a functor over a ring R", one, under good circumstances[4], simply evaluates it at $R[\varepsilon]$. In the previous section, we did exactly this, by plugging in $k_R[\varepsilon]$ to obtain the tangent space of a functor $F\colon \mathcal{C}_R \to \mathbf{Set}$. In this section, R is any $\mathcal{O}_K$-algebra and F is a 1-dimensional formal $\mathcal{O}_K$-module over R. We are interested in the deformations of F and we can define a certain deformation functor: sending an infinitesimal thickening[5] $S \twoheadrightarrow R$ to the set of certain classes of deformations of F over S.
Our main aim is to compute the tangent space of this functor, so in this section we will only be occupied with the case $S = R[\varepsilon] \to R, \varepsilon \mapsto 0$. However, we will be a bit more general than one would expect by the previous section, because R is not a field (for Schlessinger's deformation theorem one only needs to evaluate at $L[\varepsilon]$, for a certain field L). For the purposes of this thesis, this generality is indeed not necessary. Still the results one gets by being more general are useful, for a discussion of this, see Section 2.6.

Let F be a 1-dimensional formal $\mathcal{O}_K$-module law over R. Suppose $\widetilde{F}$ is a formal $\mathcal{O}_K$-module law over $R[\varepsilon]$ reducing to F modulo ε (we call $\widetilde{F}$ a *first order deformation*). Then we can write

$$\widetilde{F}(X, Y) = F(X, Y) + \varepsilon G(X, Y),$$
$$a_{\widetilde{F}}(X) = a_F(X) + \varepsilon g_a(X) \text{ for } a \in \mathcal{O}_K,$$

[4]We saw earlier that for formal groups this means R should be reduced.
[5]That is, the kernel of this map $I \subset S$ is nilpotent.

for unique power series G and g_a with coefficients in R. We can try to characterize these power series. Let us start with considering what the associativity condition on $\widetilde{F}$ implies for G. We first compute

$$\begin{aligned}\widetilde{F}(\widetilde{F}(X,Y),Z) &= F\left(F(X,Y)+\varepsilon G(X,Y),Z\right)+\varepsilon G\left(F(X,Y)+\varepsilon G(X,Y),Z\right)\\ &= F(X+_F Y,Z)+\varepsilon G(X,Y)F_1(X+_F Y,Z)+\varepsilon G(X+_F Y,Z),\end{aligned}$$

where F_1 is the partial derivative in the first component as before. Similarly, we find

$$\widetilde{F}(X,\widetilde{F}(Y,Z)) = F(X,Y+_F Z)+\varepsilon G(Y,Z)F_2(X,Y+_F Z)+\varepsilon G(X,Y+_F Z).$$

The associativity of $+_{\widetilde{F}}$ and $+_F$ now tells us that

$$G(X,Y)F_1(X+_F Y,Z)+G(X+_F Y,Z) = G(Y,Z)F_2(X,Y+_F Z)+G(X,Y+_F Z). \quad (2.6)$$

By some chain rule hocus pocus

$$F_1(X+_F Y,Z)F_1(0,X+_F Y) = F_1(0,X+_F Y+_F Z)$$

and also

$$F_2(X,Y+_F Z)F_1(0,Y+_F Z) = F_1(0,X+_F Y+_F Z). \quad (2.7)$$

Therefore, 2.6 becomes

$$\begin{aligned}&\frac{G(X,Y)}{F_1(0,X+_F Y)}F_1(0,X+_F+_Y+_F Z)+\frac{G(X+_F Y,Z)}{F_1(0,X+_F+_F Y+_F Z)}F_1(0,X+_F Y+_F Z) =\\ &\frac{G(Y,Z)}{F_1(0,Y+_F Z)}F_1(0,X+_F Y+_F Z)+\frac{G(X,Y+_F Z)}{F_1(0,X+_F Y+_F Z)}F_1(0,X+_F Y+_F Z).\end{aligned}$$

Dividing both sides by $F_1(0,X+_F Y+_F Z)$ and setting $\Delta(X,Y) \coloneqq G(X,Y)/F_1(0,X+_F Y)$ we finally obtain

$$\Delta(X,Y)+\Delta(X+_F Y,Z) = \Delta(Y,Z)+\Delta(X,Y+_F Z). \quad (2.8)$$

In the same spirit, setting $\delta_a(X) \coloneqq g_a(X)/F_1\left(0,a_F(X)\right)$, we find for all $a,b\in\mathcal{O}_K$

$$\begin{gathered}\Delta(X,Y) = \Delta(Y,X),\\ \delta_{a+b}(X) = \delta_a(X)+\delta_b(X)+\Delta\left(a_F(X),b_F(X)\right),\\ \delta_a(X)+\delta_a(Y)+\Delta\left(a_F(X),a_F(Y)\right) = a\Delta(X,Y)+\delta_a(X+_F Y),\\ \delta_{ab}(X) = a\delta_b(X)+\delta_a\left(b_F(X)\right),\end{gathered} \quad (2.9)$$

by considering commutativity, the distributivity laws, and the compatibility of the two ways of multiplying.
One can reverse these computations, and see that when the above identities hold, we retrieve a deformation $\widetilde{F}$ of F:

Proposition 2.2.1. *Let F be a 1-dimensional formal $\mathcal{O}_K$-module law over R. Let $\Delta(X,Y)$ and $\delta_a(X)$ for all $a\in\mathcal{O}_K$ be power series over R. Then*

$$\begin{cases}\widetilde{F}(X,Y) \coloneqq F(X,Y)+\varepsilon\Delta(X,Y)F_1(0,X+_F Y),\\ a_{\widetilde{F}}(X) \coloneqq a_F(X)+\varepsilon\delta_a(X)F_1\left(0,a_F(X)\right)\end{cases}$$

defines a deformation of F over $R[\varepsilon]$ if and only if the Δ and δ_a satisfy 2.8 and 2.9.

Definition 2.2.2. *An element*

$$(\Delta(X,Y),\delta_a(X))_{a\in\mathcal{O}_K}\in R[\![X,Y]\!]\times\prod_{a\in\mathcal{O}_K}R[\![X]\!]$$

satisfying 2.8 and 2.9 is called a symmetric 2-cocycle (on F)*. If it does not necessarily satisfy these equations, it is called a* symmetric 2-cochain (on F)*.*
The set of symmetric 2-cochains is naturally an R-module and the subset of symmetric 2-cocycles is a sub-R-module.

The definition above suggests we might do some cohomology soon. This is the case, although we will only stay in very low dimensions.
We have already seen that the symmetric 2-cocycles correspond to first order deformations of F. If we are going to do cohomology, we would like to mod out by "symmetric 2-coboundaries". This would result in certain isomorphism classes of deformations. Let us see what they should be.
Naively, two deformations $\widetilde{F}$ and $\widetilde{F}'$ of F could be said to be isomorphic, as soon as they are isomorphic as formal $\mathcal{O}_K$-modules. This is a bit too general for our taste however. For this we introduce the following notion:

Definition 2.2.3. *Let S be an $\mathcal{O}_K$-algebra, let $I\subset S$ be an ideal, and let $\widetilde{F},\widetilde{F}'$ be two formal $\mathcal{O}_K$-module laws over S reducing to the same formal $\mathcal{O}_K$-module law F modulo I. Then $\widetilde{F}$ and $\widetilde{F}'$ are said to be* isomorphic deformations *or* $\star$-isomorphic *deformations of F if there exist an isomorphism (called a* $\star$-isomorphism*) of formal $\mathcal{O}_K$ module laws over S*

$$f\colon\widetilde{F}\xrightarrow{\sim}\widetilde{F}'$$

that reduces to the identity modulo I.

Suppose that two first order deformations $\widetilde{F}$ and $\widetilde{F}'$, defined by the cocycles (Δ,δ_a) and (Δ',δ'_a) respectively, are $\star$-isomorphic, with $\star$-isomorphism given by

$$f(X)=X+\varepsilon g(X)\colon\widetilde{F}\to\widetilde{F}'.$$

Then

$$f(\widetilde{F}(X,Y))=F(X,Y)+\varepsilon\left(\Delta(X,Y)F_1(0,X+_FY)+g(X+_FY)\right)$$

and

$$\widetilde{F}'\left(f(X),f(Y)\right)=F(X,Y)+\varepsilon\left(F_1(X,Y)g(X)+F_2(X,Y)g(Y)+\Delta'(X,Y)F_1(0,X+_FY)\right)$$

are equal and thus

$$g(X+_FY)-F_1(X,Y)g(X)-F_2(X,Y)g(Y)=\left(\Delta'(X,Y)-\Delta(X,Y)\right)F_1(0,X+_FY).$$

Now set $\nu(X)\coloneqq g(X)/F_1(0,X)$. Dividing both sides by $F_1(0,X+_FY)$ and using the same chain rule magic 2.2 as before, we arrive at

$$\nu(X+_FY)-\nu(X)-\nu(Y)=\Delta'-\Delta.$$

Similarly, a computation yields that

$$\nu\left(a_F(X)\right)-a\nu(X)=\delta'_a-\delta_a$$

for all $a \in \mathcal{O}_K$.
Reversing the computation, one can also see that any such ν defines a $\star$-isomorphism f_ν between $\widetilde{F}$ and $\widetilde{F}'$ by the formula

$$f_\nu(X) \coloneqq X + \varepsilon\nu(X)F_1(0,X).$$

If we want the classes of $\widetilde{F}$ and $\widetilde{F}'$ to be the same, then we should mod out by $(\Delta', \delta'_a) - (\Delta, \delta_a)$, i.e., this should be a "symmetric 2-coboundary". Thus we define a *symmetric 2-coboundary (on F)* to be a symmetric 2-cochain of the form

$$(\nu(X +_F Y) - \nu(X) - \nu(Y), \nu(a_F(X)) - a\nu(X))_{a\in\mathcal{O}_K}.$$

One checks that any symmetric 2-coboundary is a cocycle.
Now we can define the *second symmetric cohomology group (of F)*[6]

$$H^2(F, \mathbb{G}_a)_s$$

to be the R-module of symmetric 2-cocycles modded out by the sub-R-module of symmetric 2-coboundaries.
From our above discussion we get:

Proposition 2.2.4 ([GH94][Prop. 7.2])**.** *Let F be a 1-dimensional formal $\mathcal{O}_K$-module law over R. Then the set of $\star$-isomorphism classes of deformations of F over $R[\varepsilon]$ is canonically isomorphic to*

$$H^2(F, \mathbb{G}_a)_s \otimes_R \operatorname{Lie} F$$

and is therefore an R-module.

Proof. From our above discussion the only thing that might be unexpected about the formulation of the proposition might be the occurrence of the Lie algebra of F.
Recall that $\operatorname{Lie} F$ is a rank 1 R-module with canonical generator D_F, where D_F is the invariant derivation dual to

$$\omega_F = \frac{1}{F_1(0,X)}dX,$$

i.e. $D_F = F_1(0,X)\dfrac{d}{dX}$. Thus we have a canonical isomorphism

$$H^2(F, \mathbb{G}_a)_s \otimes_R \operatorname{Lie} F \to \left\{\star\text{-isomorphism classes of } F \text{ over } R[\varepsilon]\right\},$$

$$\eta \coloneqq (\Delta, \delta_a) \otimes rF_1(0,X)\frac{d}{dX} \mapsto \begin{cases} F_\eta(X,Y) = F(X,Y) + \varepsilon r\Delta(X,Y)F_1(0, X +_F Y), \\ a_{F_\eta}(X) = a_F(X) + \varepsilon r\delta_a(X)F_1\left(0, a_F(X)\right), \ a \in \mathcal{O}_K \end{cases} \tag{2.10}$$

as claimed[7]. ∎

Thus, we have successfully described the "tangent space" of the $\star$-deformation class functor of F in terms of cohomology. This is an important step towards its computation.

[6]The notation with the additive group is hitherto unmotivated; in the next discussion about extensions it shall become clear that the cohomology just defined is a particular case of something more general (like every particular case).

[7]Be warned that the formula in [GH94][Prop. 7.2] differs from ours by a mistake.

Extensions

The symmetric cohomology turns out to appear in a more general context: namely, that of extensions.

Let F and F' be formal $\mathcal{O}_K$-modules over R. An *extension of F by F'* is a formal $\mathcal{O}_K$-module E together with a *short exact sequence of formal $\mathcal{O}_K$-modules*

$$0 \to F' \xrightarrow{\alpha} E \xrightarrow{\beta} F \to 0, \tag{2.11}$$

that is, we have a sequence of morphisms like above, such that for every complete adic R-algebra A, the sequence of $\mathcal{O}_K$-modules

$$0 \to F'(A) \to E(A) \to F(A) \to 0 \tag{2.12}$$

is exact.

Remark 2.2.5. *In [GH94][§6] short exact sequences are defined differently: they require that the induced sequence on the Lie algebras (or equivalently, on the invariant differentials) is exact. This is equivalent to our definition:*

1) suppose the sequence 2.11 is short exact according to our definition. We want to show

$$0 \to \operatorname{Lie} F' \xrightarrow{\alpha_*} \operatorname{Lie} E \xrightarrow{\beta_*} \operatorname{Lie} F \to 0 \tag{2.13}$$

is exact. For this, let $\mathfrak{p} \subset R$ be any prime ideal. It suffices to prove

$$0 \to R_\mathfrak{p} \otimes \operatorname{Lie} F' \to R_\mathfrak{p} \otimes \operatorname{Lie} E \to R_\mathfrak{p} \otimes \operatorname{Lie} F \to 0$$

is exact, but this is just the sequence of Lie algebras one obtains when viewing the formal $\mathcal{O}_K$-modules as defined over $R_\mathfrak{p}$. Therefore, we may assume R is local, with, say, maximal ideal $\mathfrak{m}_R$ and residue field k_R, given the discrete topology. Now, k_R is certainly reduced, whence the sequence 2.13 tensored by k_R is the same as

$$0 \to F'(k_R[\varepsilon]) \to E(k_R[\varepsilon]) \to F(k_R[\varepsilon]) \to 0,$$

which is exact by our assumption.
Since the R-modules in 2.13 are all finite free, the maps are given by matrices. We can lift a splitting $s\colon F(k_R[\varepsilon]) \to E(k_R[\varepsilon])$ to a matrix $\tilde{s}\colon \operatorname{Lie} F \to \operatorname{Lie} E$. The determinant of $\beta_* \circ \tilde{s}$ is a unit modulo $\mathfrak{m}_R$, so it is a unit in R, whence β_* is surjective. Similarly, one shows α_* is injective, and thus $\operatorname{Lie} F' \oplus \operatorname{Lie} F$ is a direct summand of $\operatorname{Lie} E$. We conclude by comparing dimensions.

2) for the other direction, assume 2.13 is exact. Choose objects $R[\![\underline{X}']\!]$, $R[\![\underline{Y}]\!]$, $R[\![\underline{X}]\!]$ representing F', E, and F respectively. Then we have a sequence corresponding to 2.11:

$$R[\![\underline{X}']\!] \leftarrow R[\![\underline{Y}]\!] \leftarrow R[\![\underline{X}]\!].$$

Choose a splitting of R-modules

$$\omega(E) \cong \omega(F') \oplus \omega(F), \;\; s\colon \omega(F') \to \omega(E).$$

By the classical description of the Lie algebra as a tangent space 1.3, we can view s as a morphism

$$s\colon (\underline{X}')/(\underline{X}')^2 \to (\underline{Y})/(\underline{Y})^2.$$

We can lift s to a continuous R-algebra morphism $R[\![\underline{X}]\!] \to R[\![\underline{Y}]\!]$, this gives us a compatible splitting

$$R[\![\underline{X}', \underline{X}]\!] \cong R[\![\underline{Y}]\!],$$

and from here on one can easily get a set-theoretic *splitting of the sequence 2.12 for every A. Thus 2.12 is exact for every A.*

It turns out that the symmetric cohomology can be generalized and that it can then be used to classify extensions.
Let F and F be formal $\mathcal{O}_K$-module laws over R again, of dimensions n and n' respectively.

Definition 2.2.6. *The* symmetric 2-cohomology on F with values in F' *is defined firstly by the following type of objects, all naturally forming $\mathcal{O}_K$-modules using the formal $\mathcal{O}_K$-module structure on F:*

- *A* symmetric 1-cochain (on F with values in F') *is a n'-tuple of power series in n variables over R without constant terms, i.e., an element of $R[\![X_1, \dots, X_n]\!]^{n'}$.*
- *A* symmetric 2-cochain *is an ordered list $(\Delta(\underline{X}, \underline{Y}), \delta_a(\underline{X}))_{a \in \mathcal{O}_K}$ of n'-tuples of power series without constant terms, where the $\underline{X}$ and $\underline{Y}$ both denote n variables.*
- *A* 2-cocycle *is a 2-cochain (Δ, δ_a) satisfying*

$$\begin{gathered}
\Delta(X, Y) = \Delta(Y, X), \\
\Delta(X, Y) +_{F'} \Delta(X +_F Y, Z) = \Delta(Y, Z) +_{F'} \Delta(X, Y +_F Z), \\
\delta_{a+b}(X) = \delta_a(X) +_{F'} \delta_b(X) +_{F'} \Delta\left(a_F(X), b_F(X)\right), \\
\delta_a(X) +_{F'} \delta_a(Y) +_{F'} \Delta\left(a_F(X), a_F(Y)\right) = a_{F'}\left(\Delta(X, Y)\right) +_{F'} \delta_a(X +_F Y), \\
\delta_{ab}(X) = a_{F'}\left(\delta_b(X)\right) +_{F'} \delta_a\left(b_F(X)\right).
\end{gathered}$$

- *The* coboundary of a 1-cochain $\nu(\underline{X}) = (\nu_1(\underline{X}), \dots, \nu_{n'}(\underline{X}))$ *is the 2-cocycle given by*

$$\delta\nu \coloneqq \begin{cases} \Delta\nu(\underline{X}, \underline{Y}) \coloneqq \nu(\underline{X} +_F \underline{Y}) -_{F'} \nu(\underline{X}) -_{F'} \nu(\underline{Y}), \\ \delta_a\nu(\underline{X}) \coloneqq \nu\left(a_F(\underline{X})\right) -_{F'} a_{F'}\left(\nu(\underline{X})\right), \ a \in \mathcal{O}_K. \end{cases}$$

The second symmetric cohomology group on F with values in F' *is the R-module $H^2(F, F')_s$ obtained by modding out the R-module of 2-cocycles by the sub-R-module of 2-coboundaries.*[8]

[8]Note that with this definition the first cohomology group will be $\mathrm{Hom}(F, F')$, so it should not surprise that the second homology group classifies extensions of F by F'.
Usually, however, it is the *zeroth* cohomology that gives $\mathrm{Hom}(F, F')$. Now one might wonder what the zeroth cohomology group is in this case, but the answer is disappointing: there is no zeroth cohomology. It would make more sense to shift all the dimensions by 1, but this is not the convention in the literature on symmetric cohomology[Mac50], therefore we do not do this.

As is clear from the definition, the previously defined cohomology was the special case where F is 1-dimensional and $F' = \mathbb{G}_a$, in which case the cohomology group turn out to even have an R-module structure.

Given an F'-valued symmetric 2-cocycle $\eta = (\Delta, \delta_a)$ on F, we can define the following extension of F by F':

$$\begin{cases} E_\eta\left((\underline{X}', \underline{X}), (\underline{Y}', \underline{Y})\right) \coloneqq (F'(\underline{X}', \underline{Y}') +_{F'} \Delta(\underline{X}, \underline{Y}), F(\underline{X}, \underline{Y}))\,, \\ a_{E_\eta}(\underline{X}', \underline{X}) \coloneqq (a_{F'}(\underline{X}') +_{F'} \delta_a(\underline{X}), a_F(\underline{X}))\,. \end{cases}$$

The cocycle relations precisely mean that the above defined E_η is a formal $\mathcal{O}_K$-module law. The formal $\mathcal{O}_K$-module E_η sits in an obvious short exact sequence

$$0 \to F' \to E_\eta \to F \to 0$$

and thus defines an extension of F by F'.
Now suppose $\eta' = (\Delta', \delta'_a)$ is another 2-cocycle, defining another extension, $E_{\eta'}$. If, η and η' are cohomologous, that is, if their difference $\eta - \eta' = \delta\nu$ is a coboundary, then we can define an morphism of formal $\mathcal{O}_K$-modules

$$f_\nu \colon E_\eta \to E_{\eta'}, (\underline{X}', \underline{X}) \mapsto (\underline{X}' -_{F'} \nu(\underline{X}), \underline{X})\,.$$

This morphism is in fact an isomorphism: indeed, $f_{-\nu}$ is its inverse. The isomorphism is even an *isomorphism of extensions*: that is, the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F' & \longrightarrow & E_\eta & \longrightarrow & F & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle f_\eta} & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & F' & \longrightarrow & E_{\eta'} & \longrightarrow & F & \longrightarrow & 0 \end{array}$$

commutes. Again, the computations are reversible, and it turns out that E_η and $E_{\eta'}$ are isomorphic extensions if and only if η and η' are cohomologous.
Let us denote the set of isomorphism classes of extensions of F by F' by $\mathrm{Ext}(F, F')$. So far, we have seen that $H^2(F, F')$ can be identified with a subset of $\mathrm{Ext}(F, F')$. This turns out to be an equality: indeed, we have essentially shown in Remark 2.2.5 that on any extension E we can choose coordinates such that E will be given by a formal $\mathcal{O}_K$-module law E_η, for some cocycle η. This gives us:

Proposition 2.2.7 ([GH94][Prop. 6.5])**.** *The symmetric F'-valued 2-cohomology group on F is in bijection with the set of isomorphism classes of extensions of F by F':*

$$H^2(F, F') \cong \mathrm{Ext}(F, F').$$

This gives $\mathrm{Ext}(F, F')$ the structure of an $\mathcal{O}_K$-module.
In the case $\dim F = 1$ and $F' = \mathbb{G}_a$, this even gives $\mathrm{Ext}(F, F')$ the structure of an R-module, and $\mathrm{Ext}(F, \mathbb{G}_a)$ classifies the first order deformation classes of F.

2.3 The height

We will be mainly interested in the "height" of the formal $\mathcal{O}_K$-modules we will deform. Therefore, let us define this concept.

Let F be a 1-dimensional formal $\mathcal{O}_K$-module law over a field extension ℓ/k. Since $\operatorname{char} k = p > 0$, we can apply the Frobenius field automorphism $x \mapsto x^p$ to the coefficients of F to obtain a new formal group law over ℓ denoted by $F^{(p)}$. We write $F^{(p^n)}$ for the n-th iterate of this construction[9].
Now let G be another 1-dimensional formal $\mathcal{O}_K$-module law over ℓ and let $f\colon F \to G$ be a morphism. Suppose $f'(0) = 0$. Then

$$0 = f'(0) \cdot \omega_F = f^* \omega_F = \omega_F\left(f(X)\right) = \frac{f'(X)}{F_1\left(0, f(X)\right)} dX.$$

Since $F_1\left(0, f(X)\right)$ is invertible, it follows that $f'(X) = 0$. Thus $f(X) = g(X^p)$ for some power series $g \in \ell[\![X]\!]$. It is not hard to see that g is a morphism of formal groups

$$F^{(p)} \to G.$$

Repeating the argument, we find that either $f = 0$ or $f(X) = g(X^{p^n})$ for some n and some g with $g'(0) \neq 0$. We claim that in the last case p^n is actually a q-power: indeed, the first non-zero coefficient of f is the one corresponding to X^{p^n}; letting $\zeta \in \mathcal{O}_K$ be a primitve $q-1$-th root of unity and remembering that $\zeta_F(X) \equiv \zeta_G(X) \equiv \zeta X \mod (X)^2$ one obtains from the identity

$$f\left(\zeta_F(X)\right) = \zeta_G\left(f(X)\right)$$

that $\zeta^{p^n} = \zeta$, i.e., the order of Frobenius in $\operatorname{Gal}(k/\mathbb{F}_p)$ divides n, so $p^n = q^h$ for some integer h.

Definition 2.3.1. *For a morphism $f\colon F \to G$ of 1-dimensional formal $\mathcal{O}_K$-modules over ℓ[10], we define the* height of f *to be the maximal $h \in Z_{\geq 0} \cup \{\infty\}$ such that there is a power series g with*

$$f(X) = g(X^h).$$

More generally, we define the height *of an endomorphism of 1-dimensional formal $\mathcal{O}_K$-modules over a complete Noetherian local ring R to be the height of the endomorphism modulo the maximal ideal $\mathfrak{m}_R$.*
We denote this by $\operatorname{ht} f \coloneqq h$. *We note that* $\operatorname{ht} f = \infty \iff f = 0$.

Definition 2.3.2. *Let F be a 1-dimensional formal $\mathcal{O}_K$-module over a complete local Noetherian $\mathcal{O}_K$-algebra R. Its* height $\operatorname{ht} F$ *is defined to be the height of the endomorphism π_F.*[11]

Example 2.3.3. *The height of the additive formal $\mathcal{O}_K$-module over ℓ is ∞.*

Example 2.3.4. *The formal $\mathbb{Z}_p$-module over $\mathbb{Z}_p$ of Example 1.3.4 is of height 1.*

[9] Note that this is in general not a formal $\mathcal{O}_K$-module, because the n-th power of Frobenius might not fix k.

[10] The index of the first non-zero coefficient of f is independent of a choice of coordinates, so this is indeed a property of morphisms between formal $\mathcal{O}_K$-modules, rather than just of formal $\mathcal{O}_K$-module *laws*.

[11] Even though we keep the uniformizer π fixed, note that the height if actually independent of the choice of uniformizer.

Note that for any $h \in \mathbb{Z}_{>0} \cup \{\infty\}$ there actually exists a formal $\mathcal{O}_K$-module over ℓ of that height: we can use the ring $\Lambda_{\mathcal{O}_K}$ to construct such a formal module. Indeed, the map

$$\Lambda_{\mathcal{O}_K} \cong \mathcal{O}_K[t_1, t_2, \dots] \to \ell, t_i \mapsto \begin{cases} 0, & \text{when } i < q^h - 1 \\ 1, & \text{when } i = q^h - 1 \\ \text{whatever}, & \text{when } i > q^h - 1 \end{cases}$$

defines a formal $\mathcal{O}_K$-module of height h.
Also note that for two 1-dimensional formal $\mathcal{O}_K$-modules F and G over ℓ of different height we have

$$\mathrm{Hom}(F, G) = 0. \tag{2.14}$$

Indeed, the height of the composition of two morphisms, is the sum of their heights. More generally, 2.14 is true when R is a complete local Noetherian $\mathcal{O}_K$-algebra, under the assumption that $\mathrm{ht}\, F < \infty$, by the following lemma:

Lemma 2.3.5 ([GH94][Lemma 4.1])**.** *Let F and G be 1-dimensional $\mathcal{O}_K$-modules over the complete Noetherian local $\mathcal{O}_K$-algebra R, where F is of height $n < \infty$. Write $\overline{F}$ and $\overline{G}$ for their reductions modulo $\mathfrak{m}_R$. Then the reduction of morphisms map*

$$\mathrm{Hom}(F, G) \to \mathrm{Hom}(\overline{F}, \overline{G})$$

is injective.

Proof. Suppose $f \colon F \to G$ is a morphism with coefficients in $\mathfrak{m}_R$. By induction on n we prove f has coefficients in $\mathfrak{m}_R^n$. The case $n = 1$ is true by assumption. Now suppose it has been proven that f has coefficients in $\mathfrak{m}_R^n$ for some $n \geq 1$. Then $\pi_g \circ f$ has coefficients in $\mathfrak{m}_R^{n+1}$, that is, $f \circ \pi_F$ has coefficients in $\mathfrak{m}_R^{n+1}$. Since F has finite height, we see this implies that f itself lies in $\mathfrak{m}_R^{n+1}$.
We have proven by induction that f has coefficients in the intersection of all powers of the maximal ideal of R, and therefore equals 0. ∎

Over ℓ we have a nice characterisation of infinite height formal $\mathcal{O}_K$-modules:

Proposition 2.3.6 ([Dri74][Prop. 1.6])**.** *Let F be a 1-dimensional formal $\mathcal{O}_K$-module over ℓ. Then*

$$\mathrm{ht}\, F = \infty \iff F \cong \mathbb{G}_a.$$

Before we give the prove, let us state the following technical lemma, whose proof is an immediate computation. It will allow us to inductively construct the isomorphism of the above lemma.

Lemma 2.3.7. *Let $n \geq 1$ and F a 1-dimensional formal $\mathcal{O}_K$-module law over an $\mathcal{O}_K$-algebra R. Suppose $f \in R[\![X]\!]$ is equal to $X - rX^n +$ higher order terms then*

$$f\left(F\left(f^{-1}(X), f^{-1}(Y)\right)\right) \equiv F(X, Y) - r\left((X+Y)^n - X^n - Y^n\right) \mod \text{degree } n+1,$$
$$f\left(a_F\left(f^{-1}(X)\right)\right) \equiv a_F(X) - r(a^n - a)X^n \mod \text{degree } n+1,$$

for all $a \in \mathcal{O}_K$.

Proof of Proposition 2.3.6. The right to left direction is immediate.
Assume F is of infinite height. Assume $(F, a_F) \equiv (\mathbb{G}_a, a_{\mathbb{G}_a})$ mod degree n. By Corollary 1.4.5, if n is not a q-power, we can do, by means of the above lemma, a change of variables on F such that $(F, a_F) \equiv (\mathbb{G}_a, a_{\mathbb{G}_a})$ mod degree $n+1$ after this change of variables. If n is a q-power, it follows from Proposition 1.4.3 that $(F, a_F) \equiv (\mathbb{G}_a, a_{\mathbb{G}_a})$ mod degree $n+1$ already.
This way, we can inductively construct the sought after isomorphism. ■

Remark 2.3.8. *Note that in exactly the same way one shows that on a 1-dimensional formal $\mathcal{O}_K$-module F over ℓ of height h one can choose coordinates such that*

$$(F, a_F) \equiv (\mathbb{G}_a, a_{\mathbb{G}_a}) \mod \text{degree } q^h.$$

Corollary 2.3.9. *Let R be a k-algebra and F a "1-dimensional formal k-module over R" (definition obvious), then F is isomorphic to the additive "formal k-module".*

Proof. Indeed, k is a $k[\![\pi]\!]$-algebra by sending π to 0 and F is then a formal $k[\![\pi]\!]$-module in the obvious way, of height ∞ and therefore additive. ■

Definition 2.3.10. *A 1-dimensional formal $\mathcal{O}_K$-module law F over ℓ of height h is called* normal *if $\pi_F(X) = X^{q^h}$ and*

$$F(X,Y) \equiv X + Y,\ a_F(X) \equiv aX \mod \text{degree } q^h$$

for all $a \in \mathcal{O}_K$.

Remark 2.3.11. *A normal height h formal $\mathcal{O}_K$-module law F has coefficients in $\mathbb{F}_{q^h}$: indeed, F and the a_F all commute with $\pi_F(X) = X^{q^h}$.*

Proposition 2.3.12. *Every height h formal $\mathcal{O}_K$-module F over a separably closed ℓ is isomorphic to a normal one.*

Proof. We know $\pi_F(X) \equiv aX^{q^h}$ mod degree q^h+1 with $a \neq 0$. Taking $b \in \ell$ with $a = b^{q^h-1}$, then after the change of variables $T = bX$ we may assume $a = 1$.
Then

$$\pi_F(T) = T^{q^h} + cT^{kq^h} + \text{higher order terms.}$$

For $e \in \ell$ set $\varphi_e(T) \coloneqq T - eT^k$ we have

$$\varphi_e\left(\pi_F\left(\varphi_e^{-1}(T)\right)\right) = T^{q^h} + (e^{q^h} - e + c)T^{q^h k} \mod \text{degree } q^h k + 1. \tag{2.15}$$

Letting $e \in \ell$ be a root of the separable Artin-Schreier polynomial $X^{q^h} - X + c$, we may assume, after a change of variables by φ_e, that $c = 0$. Therefore, in good coordinates we have $\pi_F(T) = T^{q^h}$.
We have already ensured with these coordinates, that the coefficients of F and the a_F are in $\mathbb{F}_{q^h}$. Therefore, if we now use the change of variables of Proposition 2.3.6 to ensure that

$$(F, a_F) \equiv (\mathbb{G}_a, a_{\mathbb{G}_a}) \mod \text{degree } q^h,$$

we will still have π_F of the required form, by the computation 2.15. ■

Proposition 2.3.13 ([Dri74][Prop. 1.7])**.** *Two height h formal $\mathcal{O}_K$-modules F and G over a separably closed ℓ are isomorphic.*

Proof. By the preceding proposition we may assume both are normal. We can use Lemma 2.3.7 to inductively construct an isomorphism $f \in \mathbb{F}_{q^h}[\![X]\!]$. Note again that these kind of isomorphisms (with $\mathbb{F}_{q^h}$-coefficients) respect normality.
We take care of the degree not a q-power as in the proof of Proposition 2.3.6. If now n is a q-power and

$$(F, a_F) \equiv (G, a_G) \mod \text{degree } n,$$

it follows from Proposition 1.4.3 and the fact that $\pi_F(X) = \pi_G(X)$ that

$$(F, a_F) \equiv (G, a_G) \mod \text{degree } n+1,$$

so this case is automatically taken care of. ■

2.4 The Lubin-Tate deformation functor

Now, we can finally define the Lubin-Tate deformation functor.
Let $\overline{F}$ be a 1-dimensional formal $\mathcal{O}_K$-module law over k of height h satisfying

$$(\overline{F}, a_{\overline{F}}) \equiv (\mathbb{G}_a, a_{\mathbb{G}_a}) \mod \text{degree } q^h$$

and

$$\pi_{\overline{F}}(X) = X^{q^h} \mod \text{degree } q^h + 1. \tag{2.16}$$

Then, under the assumption of Remark 1.4.4 that $\mathcal{O}_K[t_1, t_2, \dots] \xrightarrow{\sim} \Lambda_{\mathcal{O}_K}$ sends t_{n-1} to π_n for q-power n, we see that $\overline{F}$ corresponds to a morphism

$$f_{\overline{F}} \colon \mathcal{O}_K[t_1, t_2, \dots] \to k \tag{2.17}$$

that sends t_i to 0 whenever $i < q^h - 1$ and t_{q^h-1} to 1.
We let $\mathcal{C}$ be the category of complete local Noetherian $\mathcal{O}_K$-algebras, where the morphisms are continuous (and thus local) $\mathcal{O}_K$-algebra morphisms. Note that for any $R \in \mathcal{C}$ the category $\mathcal{C}_R$ defined in Section 2.1 is a full subcategory of $\mathcal{C}$.
We define the deformation functor corresponding to $\overline{F}$ by

$$\mathcal{D}\mathrm{ef} \colon \mathcal{C} \to \mathbf{Set}, R \mapsto \{\star\text{-isomorphism classes of deformations } \widetilde{F} \text{ of } \overline{F} \text{ over } R\}.$$

Note that in this definition it is understood that $\widetilde{F}$ equals $\overline{F}$ modulo $\mathfrak{m}_R$, the maximal ideal of R.
Our aim is to prove that $\mathcal{D}\mathrm{ef}$ is represented by the complete Noetherian local $\mathcal{O}_K$-algebra

$$\mathcal{O}_K[\![u_1, \dots u_{h-1}]\!].$$

To prove this, we will show that if $\widetilde{F}$ is a deformation of $\overline{F}$ over R, the R-module $H^2(\widetilde{F}, \mathbb{G}_a)_s$ of first order deformation classes over $R[\varepsilon]$ is free of rank $h-1$. In particular, for $R = k_R$ we obtain that the tangent space of $\mathcal{D}\mathrm{ef}\,|_{\mathcal{C}_R}$ over k_R is $h-1$-dimensional. It then follows from Schlessinger's deformation theorem that

$$\mathrm{Hom}(\mathcal{O}_K[\![u_1, \dots, u_{h-1}]\!], -)|_{\mathcal{C}_R} = \mathrm{Hom}(R[\![u_1, \dots, u_{h-1}]\!], -) \cong \mathcal{D}\mathrm{ef}\,|_{\mathcal{C}_R},$$

here the first homset is of continuous $\mathcal{O}_K$-algebra morphisms and the second of continuous R-algebra morphisms.

From the above equation it is not hard to conclude that $\mathcal{D}\mathrm{ef}$ restricted to the category of local Artinian $\mathcal{O}_K$-algebras is prorepresented by $\mathcal{O}_K[\![u_1,\dots,u_{h-1}]\!]$. Then one concludes with the observation that $R/\mathfrak{m}_R^n$ is local Artinian, for any n, and that

$$\begin{aligned}\mathrm{Hom}(\mathcal{O}_K[\![u_1,\dots,u_{h-1}]\!],R) &\cong \varprojlim_n \mathrm{Hom}(\mathcal{O}_K[\![u_1,\dots,u_{h-1}]\!],R/\mathfrak{m}_R^n)\\ &= \varprojlim_n \mathcal{D}\mathrm{ef}(R/\mathfrak{m}_R^n)\\ &\cong \mathcal{D}\mathrm{ef}(R),\end{aligned} \tag{2.18}$$

where the last step requires a small argument (Claim 2.4.3).

Computation of $H^2(\widetilde{F},\mathbb{G}_a)_s$

Theorem 2.4.1 ([LT66][Prop. 2.6], [GH94][Prop. 9.8])**.** *Let $\widetilde{F}$ be 1-dimensional formal $\mathcal{O}_K$-module over the complete local Noetherian $\mathcal{O}_K$-algebra R, such that $\widetilde{F}$ mod $\mathfrak{m}_R$ is equal to $\overline{F}$. Then the symmetric 2-cohomology $H^2(\widetilde{F},\mathbb{G}_a)_s$ is free over R of rank $h-1$ and has a basis represented by cocycles $(\Delta_i,\delta_{a,i})$ for $i\in\{1,\dots,h-1\}$ satisfying*

$$\delta_{\pi,i}\equiv X^{q^i}+(\textit{higher order terms}) \mod \mathfrak{m}_R \textit{ for } i\in\{0,\dots,h-1\}. \tag{2.19}$$

Moreover, if $(\Delta_1,\delta_{a,1}),\dots,(\Delta_{h-1},\delta_{a,h-1})$ are cocycles satisfying the above equation, then they represent a basis.

Proof. Let us first explain how the cocycles are obtained.
Let $f_{\widetilde{F}}\colon \mathcal{O}_K[t_1,t_2,\dots]\to R$ be the morphism corresponding to $\widetilde{F}$ (we are still making the assumption of Remark 1.4.4). For each i we deform this to a morphism $f_{\widetilde{F}_i}\colon \mathcal{O}_K[t_1,t_2,\dots]\to R[\varepsilon]$ by

$$f_{\widetilde{F}^i}\colon t_j\mapsto\begin{cases} f_{\widetilde{F}}(t_j)+\varepsilon, & \text{if } j=q^i-1\\ f_{\widetilde{F}}(t_j), & \text{else}\end{cases}.$$

These morphisms correspond to first order deformations $\widetilde{F}^i$ of $\widetilde{F}$, which in turn correspond to cocycles $(\Delta_i,\delta_{a,i})$, for $i\in\{1,\dots,h-1\}$. By 2.10 we have the following formula:

$$\delta_{\pi,i}(X)\varepsilon=\frac{\pi_{\widetilde{F}^i}(X)-\pi_{\widetilde{F}}(X)}{\widetilde{F}_1\left(0,\pi_{\widetilde{F}}(X)\right)}.$$

Since $\widetilde{F}$ reduces to $\overline{F}$ modulo $\mathfrak{m}_R$ we see that

$$\pi_{\widetilde{F}^i}(X)-\pi_{\widetilde{F}}(X)\equiv\varepsilon X^{q^i}+(\text{higher order terms}) \mod \mathfrak{m}_R. \tag{2.20}$$

Therefore, we conclude that for all $i\in\{1,\dots,h-1\}$

$$\delta_{\pi,i}\equiv X^{q^i}+(\text{higher order terms}) \mod \mathfrak{m}_R.$$

It remains to show that when $(\Delta_1,\delta_{a,1}),\dots,(\Delta_{h-1},\delta_{a,h-1})$ satisfy 2.19 they form a basis. Let us first show they span.

We make the following observation. Let $\eta = (\Delta, \delta_a)$ is a cocycle and $\ell, m \geq 1$ (they might be ∞), such that

$$\eta \equiv 0 \mod \mathfrak{m}_R^m, \text{degree } \ell.$$

Then we consider the formal $\mathcal{O}_K$-module $\widetilde{F}_\eta$ over $R/\mathfrak{m}_R^m$ defined by

$$\begin{cases} \widetilde{F}_\eta(X,Y) = \widetilde{F}(X,Y) + \varepsilon\Delta(X,Y)\widetilde{F}_1(0, X +_{\widetilde{F}} Y) \mod \mathfrak{m}_R^m, \\ a_{\widetilde{F}_\eta}(X) = a_{\widetilde{F}}(X) + \varepsilon\delta_a(X)\widetilde{F}_1\left(0, a_{\widetilde{F}}(X)\right) \mod \mathfrak{m}_R^m,\ a \in \mathcal{O}_K. \end{cases}$$

We apply Corollary 1.4.5 to $\widetilde{F}_\eta$ and $\widetilde{F}$ to find that:

$$\begin{aligned} \Delta(X,Y) \equiv \Delta(X,Y)\widetilde{F}_1(0, X +_{\widetilde{F}} Y) \equiv c\left((X+Y)^n - X^n - Y^n\right) \mod \mathfrak{m}_R^m, \text{degree } \ell+1, \\ \delta_a(X) \equiv \delta_a(X)\widetilde{F}_1\left(0, a_{\widetilde{F}}(X)\right) \equiv c(a^n - a)X^n \mod \mathfrak{m}_R^m, \text{degree } \ell+1, \end{aligned} \tag{2.21}$$

for some $c \in R$, when ℓ is not a q-power; and, when ℓ is a q-power

$$\begin{aligned} \Delta(X,Y) \equiv \Delta(X,Y)\widetilde{F}_1(0, X +_{\widetilde{F}} Y) \equiv c\frac{p}{\pi}C_\ell(X,Y) \mod \mathfrak{m}_R^m, \text{degree } \ell+1, \\ \delta_a(X) \equiv \delta_a(X)\widetilde{F}_1\left(0, a_{\widetilde{F}}(X)\right) \equiv c\frac{(a^\ell - a)}{\pi}X^\ell \mod \mathfrak{m}_R^m, \text{degree } \ell+1, \end{aligned} \tag{2.22}$$

for some $c \in R$ again.
Now one checks the following computation of coboundaries, modulo the maximal ideal and higher order terms, for all $n \geq 1, j \geq 0$:

$$\begin{aligned} \delta X^n &\equiv \left((X+Y)^n - X^n - Y^n, (a^n - a)X^n\right)_{a \in \mathcal{O}_K} \mod \mathfrak{m}_R, \text{degree } n+1, \\ \delta X^{q^j} &\equiv c_j \left(\tfrac{\pi}{p}C_{q^{j+h}}(X,Y), \tfrac{a^{q^{j+h}} - a}{\pi}X^{q^{j+h}}\right)_{a \in \mathcal{O}_K} \mod \mathfrak{m}_R, \text{degree } q^{j+h}+1, \end{aligned} \tag{2.23}$$

for certain $c_j \notin \mathfrak{m}_R$; the first equation is immediate and the second follows from the facts that X^{q^j} is an endomorphism of $\mathbb{G}_a$ and that $\widetilde{F} \mod \mathfrak{m}_R$ is additive up to degree $q^h - 1$ from which it follows that

$$\delta X^{q^j} \equiv 0 \mod \mathfrak{m}_R, \text{degree } q^{h+j};$$

from here we can use 2.22 with $\ell = q^{h+j}$; finally, to see that the coefficient c_j is a unit, observe that

$$\delta_\pi X^{q^j} \equiv \left(\pi_{\widetilde{F}}(X)\right)^{q^j} - \pi X^{q^j} \equiv X^{q^{h+j}} \mod \mathfrak{m}_R, \text{degree } q^{h+j}+1.$$

We are ready to prove the classes of the $\eta_i := (\Delta_i, \delta_{a,i})$ span the 2-cohomology. Let $\eta = (\Delta, \delta_a)$ be any cocycle now. Inductively on the degree, we can construct, by 2.19 and 2.23, an infinite linear combination of the η_i's, δX^n's, and δX^{q^j}'s whose difference, η', with η has coefficients in $\mathfrak{m}_R$. Then, again inductively on the degree and by 2.19 and 2.23, we construct an infinite linear combination of the $\eta_i, \delta X^n, \delta X^{q^j}$, *with coefficients in* $\mathfrak{m}_R$, whose difference η'' with η' has coefficients in $\mathfrak{m}_R^2$. Going on like this, we find in the limit an infinite linear combination of η_i and coboundaries that is equal to η:

$$(\eta - \eta') + (\eta' - \eta'') + \cdots = \eta.$$

This proves that the classes of the η_i span.
Finally, we prove independence of the η_i. Our description 2.23 of coboundaries modulo $\mathfrak{m}_R$ and 2.19 shows that if a linear combination

$$\sum_i \lambda_i \eta_i$$

is a coboundary, then all λ_i belong to $\mathfrak{m}_R$. Then it follows that these coboundaries themselves must have coefficients in $\mathfrak{m}_R$. From 2.23 and 2.19 again, it must then be that the λ_i belong to $\mathfrak{m}_R^2$. Et cetera... Thus all λ_i are 0. This finishes the proof. ■

$\mathcal{D}\mathrm{ef}$ is represented by $\mathcal{O}_K[\![u_1, \ldots, u_{h-1}]\!]$

Let R be any complete Noetherian local $\mathcal{O}_K$-algebra.

Proposition 2.4.2. *The functor $\mathcal{D}\mathrm{ef}\,|_{\mathcal{C}_R}$ satisfies the Mayer-Vietoris property.*

Proof. Indeed, let $A \to C \leftarrow B$ be a diagram in $\mathcal{C}_R$. We must show that the natural morphism

$$\mathcal{D}\mathrm{ef}(A \times_C B) \to \mathcal{D}\mathrm{ef}(A) \times_{\mathcal{D}\mathrm{ef}(C)} \mathcal{D}\mathrm{ef}(B)$$

that sends the class of a deformation $\widetilde{F}$ over $A \times_C B$ to the pair of classes of the base-changes $\mathrm{pr}_{1*}\widetilde{F}$ and $\mathrm{pr}_{2*}\widetilde{F}$, to A and B respectively (so $\mathrm{pr}_1 \colon A \times_C B \to A$ and $\mathrm{pr}_2 \colon A \times_C B \to B$ are the projections).
First, let us prove injectivity. Suppose two deformations $\widetilde{F}$ and $\widetilde{F}'$ over $A \times_C B$ are sent to the same pair. Then we have $\star$-isomorphisms $\varphi_i \colon \mathrm{pr}_{i*}\widetilde{F} \to \mathrm{pr}_{i*}\widetilde{F}'$ for $i \in \{1, 2\}$. After base-change to C they become the same, because by Lemma 2.3.5 there is at most one $\star$-deformation between the base-changes of $\widetilde{F}$ and $\widetilde{F}'$ over C. Therefore $\varphi_1 \times \varphi_2$ is a well-defined $\star$-isomorphism from $\widetilde{F} = (\mathrm{pr}_{1*}\widetilde{F}, \mathrm{pr}_{2*}\widetilde{F})$ to $\widetilde{F}' = (\mathrm{pr}_{1*}\widetilde{F}', \mathrm{pr}_{2*}\widetilde{F}')$ over $A \times_C B$. This proves injectivity.
To prove surjectivity, let $\widetilde{F}_1$ and $\widetilde{F}_2$ be deformations over A and B, respectively, that give the same deformation class over C. Let $\varphi \colon \widetilde{F}_1 \to \widetilde{F}_2$ be a $\star$-isomorphism over C. We can lift it to a $\varphi'(X) \in XA[\![X]\!]$ and we may assume F_1 equals $\varphi'\big(\widetilde{F}_1\,(\varphi'^{-1}(X), \varphi'^{-1}(Y))\,\big)$. But then $\widetilde{F}_1$ and $\widetilde{F}_2$ are the same over C, whence they glue to a deformation $\widetilde{F}$ over $A \times_C B$. This proves surjectivity. ■

Formal smoothness of the deformation functor follows from the description of $\Lambda_{\mathcal{O}_K}$ as a polynomial ring. Therefore, Theorem 2.4.1 applied to k_R shows that

$$\mathrm{Hom}(R[\![u_1, \ldots, u_{h-1}]\!], -) \cong \mathcal{D}\mathrm{ef}\,|_{\mathcal{C}_R}$$

by Schlessinger's deformation theorem.

Claim 2.4.3. *For any complete Noetherian local $\mathcal{O}_K$-algebra S, we have*

$$\mathcal{D}\mathrm{ef}(S) \cong \varprojlim_n \mathcal{D}\mathrm{ef}(S/\mathfrak{m}_S^n).$$

Proof. Suppose we have, for every $n \geq 1$, $\star$-isomorphic deformations $\widetilde{F}_n$ and $\widetilde{F}'_n$ of $\overline{F}$ over $S/\mathfrak{m}_S^n$ such that

$$(\widetilde{F}_{n+1}, a_{\widetilde{F}_{n+1}}) \equiv (\widetilde{F}_n, a_{\widetilde{F}_n}),\ (\widetilde{F}'_{n+1}, a_{\widetilde{F}'_{n+1}}) \equiv (\widetilde{F}'_n, a_{\widetilde{F}'_n}) \mod \mathfrak{m}_S^n.$$

Our claim is proven if we can show that the deformations "$\lim_{n\to\infty} \widetilde{F}_n$" and "$\lim_{n\to\infty} \widetilde{F}'_n$" of $\overline{F}$ over S are $\star$-isomorphic as well. This is true: Lemma 2.3.5 implies the $\star$-isomorphisms between the $\widetilde{F}_n$ and $\widetilde{F}'_n$ are all unique. Hence, they are all compatible and can be lifted to a $\star$-isomorphism over S. ∎

It now follows from 2.18 that $R[\![u_1, \ldots, u_{h-1}]\!]$ represents $\mathcal{D}\mathrm{ef}$ restricted to the category of complete Noetherian local R-algebras with residue field k_R. In particular, by the Yoneda lemma, there exists a universal $\star$-isomorphism class of a deformation of $\overline{F}$ over R. Let $\widetilde{F}_R$ be a representative of this class.
Remember that $\overline{F}$ comes from the morphism $f_{\overline{F}}$ as defined in 2.17. In Theorem 2.4.1 we have seen that, for $i \in \{1, \ldots, h-1\}$, the deformations F^i of $\overline{F}$ over $k[\varepsilon]$, corresponding to the maps

$$f_{F^i} \colon t_j \mapsto \begin{cases} f_{\overline{F}}(t_j) + \varepsilon, & \text{if } j = q^i - 1 \\ f_{\overline{F}}(t_j), & \text{else} \end{cases} \tag{2.24}$$

span the tangent space of the deformation functor $\mathcal{D}\mathrm{ef}$. To be more precise: they span the tangent space over k_R of the restricted functor $\mathcal{D}\mathrm{ef}\,|_{\mathcal{C}_R}$, for any $R \in \mathcal{C}$.
Since $\widetilde{F}_R$ is universal, there is an endomorphism φ of $R[\![\underline{u}]\!] \coloneqq R[\![u_1, \ldots, u_{h-1}]\!]$ such that $\varphi_* \widetilde{F}_R$ equals $\widetilde{F}_{\mathcal{O}_K}$, which is the universal deformation over $\mathcal{O}_K[\![\underline{u}]\!]$. (We view $\widetilde{F}_{\mathcal{O}_K}$ as an element of $R[\![\underline{u}]\!]$ via the canonical morphism $\mathcal{O}_K[\![\underline{u}]\!] \to R[\![\underline{u}]\!]$.) And because the F^i are specializations of $\widetilde{F}_{\mathcal{O}_K}$, it follows that φ induces a surjection on the tangent spaces over k_R. This surjection is an endomorphism of the tangent space, which is of finite dimension, and thus an isomorphism. Therefore, φ must be an isomorphism as well.
For any complete local Noetherian R-algebra A with residue field k_R, the composition

$$\mathrm{Hom}(\mathcal{O}_K[\![\underline{u}]\!], A) = \mathrm{Hom}(R[\![\underline{u}]\!], A) \xrightarrow{-\circ\varphi} \mathrm{Hom}(R[\![\underline{u}]\!], A) \cong \mathcal{D}\mathrm{ef}(A)$$

is an isomorphism sending a morphism $f \colon \mathcal{O}_K[\![\underline{u}]\!] \to A$ to the $\star$-isomorphism class of the deformation

$$(f\varphi)_* \widetilde{F}_R = f_* \widetilde{F}_{\mathcal{O}_K}.$$

It follows that the unrestricted deformation functor $\mathcal{D}\mathrm{ef}$ from to category of complete local Noetherian $\mathcal{O}_K$-algebras to the category of sets is represented by

$$\mathcal{O}_K[\![u_1, \ldots, u_{h-1}]\!].$$

That is,

$$\mathcal{D}\mathrm{ef} \cong \mathrm{Spf}\, \mathcal{O}_K[\![u_1, \ldots, u_{h-1}]\!]$$

and we have a *universal deformation* $F_{\mathrm{univ}} \coloneqq \widetilde{F}_{\mathcal{O}_K}$ of $\overline{F}$ over $\mathcal{O}_K[\![u_1, \ldots, u_{h-1}]\!]$.
The following will be useful in the future:

Proposition 2.4.4. *The coefficients of $X_1, \ldots, X^{q^h-1}$ in $\pi_{F_{univ}}$ generate the maximal ideal of $\mathcal{O}_K[\![u_1, \ldots, u_{h-1}]\!]$.*

Proof. The maximal ideal is $(\pi, u_1, \ldots, u_{h-1})$, so we need to prove the ideal generated by the coefficients contains each u_i. We know that the first order deformations F^i from 2.24 span $H^2(\overline{F}, \mathbb{G}_a)_s$, the tangent space to $\mathcal{D}\mathrm{ef}$.
From 2.16 and 2.20 it follows that

$$\pi_{F^i}(X) = \varepsilon X^{q^i} + (\text{higher order terms}).$$

By the proof of Schlessinger's deformation theorem the deformation F^i corresponds to the morphism $\mathcal{O}_K[\![u_1, \ldots, u_{h-1}]\!] \to k[\varepsilon],\ u_j \mapsto \begin{cases} \varepsilon, & \text{when } i = j \\ 0, & \text{else} \end{cases}$. Since the F^i are all specializations of $F - \mathrm{univ}$ it follows that the u_i are contained among the coefficients modulo $(\pi, u_1, \ldots, u_{h-1})^2$. The result follows by an approximation argument. ∎

The formal spectrum $\operatorname{Spf} \mathcal{O}_K[\![u_1, \ldots, u_{h-1}]\!]$ is called the *(level 0) Lubin-Tate space associated to* $\overline{F}$ and is also written as

$$\mathcal{M}_{\overline{F},0},$$

or, when $\overline{F}$ is understood, simply as

$$\mathcal{M}_0.$$

In the next Chapter we will define the higher level Lubin-Tate spaces.

2.5 $\mathcal{O}_K$-typical modules

This section and the next are "bonus" sections, that is, they are not necessary for the next chapter. The reader who is only interested in the higher levels of Lubin-Tate spaces is recommended to go straight to the final chapter.

In [GH94][§5] a specific universal deformation is given that has nice computational properties, this formal $\mathcal{O}_K$-module they call an "$\mathcal{O}_K$-typical module". It generalizes a nice normal form which can be obtained from Corollary 2.3.9: namely, any 1-dimensional formal $k[\![\pi]\!]$-module F is, as a "formal k-module" isomorphic to the additive group, and thus has the following form under good coordinates:

$$\begin{aligned} F(X, Y) &= X + Y, \\ \zeta_F(X) &= \zeta X,\ \zeta \in k, \\ \pi_F(X) &= \pi X + \sum_{i>0} a_i X^{q^i}. \end{aligned} \tag{2.25}$$

In mixed characteristic, life is a bit harder, but theory from [Haz78] comes to the rescue.

Hazewinkel's universal $\mathcal{O}_K$-typical module

Let f be the unique power series with constant term and with coefficients in $K[\underline{v}] = K[v_1, v_2, \ldots]$ satisfying the "functional equation"

$$f(X) = X + \sum_{i>0} \frac{v_i}{\pi} f^{(q^i)}(X^{q^i}),$$

where we adopt the convention that for a power series g with coefficients in $K[\underline{v}]$ the power series $g^{(q^i)}$ is the power series obtained from g by replacing each v_j by $v_j^{q^i}$. For any $q-1$-th root of unity ζ one has $f(\zeta X) = \zeta f(X)$, and thus the series is of the form

$$f(X) = \sum_i b_i X^{q^i}.$$

By induction one proves the following formula for the coefficients:

$$b_k = \sum_{\substack{0 \le \ell \le k, \\ (a_1,\dots,a_\ell) \in \mathbb{Z}_{\ge 1}^{\ell}, \\ \sum a_i = k}} \prod_{j=1}^{\ell} \pi^{-1} v_{a_j}^{q^{\sum_{i<j} a_i}}, \tag{2.26}$$

which is a horrible formula but at least allows us to see that

$$f(X) \equiv X + \frac{v_k}{\pi} X^{q^k} + (\text{higher order terms}) \mod (v_1, \dots, v_{k-1}) \tag{2.27}$$

and that $\pi^k b_k \in \mathcal{O}_K[\underline{v}]$.
We set

$$\begin{cases} F(X,Y) \coloneqq f^{-1}\left(f(X) + f(Y)\right), \\ a_F(X) \coloneqq f^{-1}\left(af(X)\right), \ a \in \mathcal{O}_K, \end{cases} \tag{2.28}$$

which defines a 1-dimensional formal $\mathcal{O}_K$-module law over $K[\underline{v}]$. By construction, its logarithm is f. It turns out that:

Proposition 2.5.1. *The 1-dimensional formal $\mathcal{O}_K$-module law 2.28 has coefficients in $\mathcal{O}_K[v_1, v_2, \dots]$.*

Proof. Hazewinkel proves this for f satisfying general "functional equations", see [Haz78][§2]. In our less general case, the proof is simpler, at least in terms of notation. Let us show F has integral coefficients, and leave the case of the a_F to the reader.
Since $f(\zeta X) = \zeta f(X)$ for all $q-1$-th roots of unity $\zeta \in \mathcal{O}_K$, the same is true for the inverse f^{-1}. Therefore $F(\zeta X, \zeta Y) = \zeta F(X,Y)$, and thus all the non-zero homogeneous parts of F are of q-power degree.
Write

$$F(X,Y) = D_0(X,Y) + D_1(X,Y) + D_2(X,Y) + \cdots,$$

for the decomposition of F into its degree q^i homogeneous parts D_i. By induction on n we will show that the D_n has integral coefficients. The case $n = 0$ is easy.
Suppose $n > 0$ and that the D_i are integral for $i < n$.
Firstly, note that for $r > 1$ we have

$$F^r = (D_0 + \cdots + D_{n-1})^r \mod \text{degree } q^n + 1. \tag{2.29}$$

Secondly, not that for $i, j \ge 0$, we have, by the classical

$$x \equiv y \mod p^k \implies x^p \equiv y^p \mod p^{k+1},$$

that

$$\begin{aligned} &(D_0(X,Y) + \cdots + D_{n-1}(X,Y))^{q^{i+j}} \equiv \\ &\left(D_0^{(q^j)}(X^{q^j}, Y^{q^j}) + \cdots + D_{n-1}^{(q^j)}(X^{q^j}, Y^{q^j})\right)^{q^i} \mod \pi^{i+1}\mathcal{O}_K[\underline{v}]. \end{aligned} \tag{2.30}$$

Combining 2.29 and 2.30 gives us, for $i \geq 0$ and $j \geq 1$ that

$$F(X,Y)^{q^{i+j}} \equiv F^{(q^j)}(X^{q^j}, Y^{q^j})^{q^i} \mod \pi^{i+1}\mathcal{O}_K[\underline{v}], \text{degree } q^n + 1. \tag{2.31}$$

Now consider

$$f(X) + f(Y) = f\left(F(X,Y)\right) = F(X,Y) + \sum_{j\geq 1} \frac{v_j}{\pi} \sum_{i \geq 0} b_i^{(q^j)} F(X,Y)^{q^{i+j}},$$

where the last step is by the "functional equation".
Since, as discussed above, $\pi^i b_i \in \mathcal{O}_K[\underline{v}]$ for every i, we get from 2.31 that

$$\frac{v_j}{\pi} b_i^{(q^j)} F(X,Y)^{q^{i+j}} \equiv \frac{v_j}{\pi} b_i^{(q^j)} F^{(q^j)}(X^{q^j}, Y^{q^j})^{q^i} \mod \mathcal{O}_K[\underline{v}], \text{degree } q^n + 1.$$

This gives

$$\begin{aligned} f(X) + f(Y) &\equiv F(X,Y) + \sum_{j\geq 1} \frac{v_j}{\pi} \sum_{i \geq 0} b_i^{(q^j)} F^{(q^j)}(X^{q^j}, Y^{q^j})^{q^i} \\ &\equiv F(X,Y) + \sum_{j \geq 1} \frac{v_j}{\pi} f^{(q^j)}\left(F^{(q^j)}(X^{q^j}, Y^{q^j})\right) \mod \mathcal{O}_K[\underline{v}], \text{degree } q^n + 1. \end{aligned}$$

Finally, notice that $f\left(F(X,Y)\right) = f(X) + f(Y)$ implies

$$f^{(q^j)}\left(F^{(q^j)}(X,Y)\right) = f^{(q^j)}(X) + f^{(q^j)}(Y).$$

Therefore, using the "functional equation" in the last second to last step, we find

$$\begin{aligned} f(X) + f(Y) &\equiv F(X,Y) + \sum_{j \geq 1} \frac{v_j}{\pi}\left(f^{(q^j)}(X^{q^j}) + f^{(q^j)}(Y^{q^j})\right) \\ &\equiv F(X,Y) + f(X) + f(Y) - X - Y \\ &\equiv f(X) + f(Y) + F(X,Y) \mod \mathcal{O}_K[\underline{v}], \text{degree } q^n + 1. \end{aligned}$$

Thus, we see that $F(X,Y)$ has integral coefficients up to degree q^n, from which we obtain integrality of D_n, which is what we wanted to show. ■

The formal $\mathcal{O}_K$-module law F is called the *universal $\mathcal{O}_K$-typical module*. This is because we define an *$\mathcal{O}_K$-typical module over an $\mathcal{O}_K$-algebra R* to be a specialization of F via a morphism $\mathcal{O}_K[\underline{v}] \to R$ (and this morphism is unique, [Haz78][§21.5.5]). Thus, F is universal by definition of $\mathcal{O}_K$-typical modules.
There is also a better reason to call F "universal": every 1-dimensional formal $\mathcal{O}_K$-module happens to have $\mathcal{O}_K$-typical coordinates, see [Haz78][§21.5.6].

The Lubin-Tate space associated to an $\mathcal{O}_K$-typical module over k

Recall equation 2.27: $f(X) \equiv X + \frac{v_k}{\pi} X^{q^k} + (\text{higher order terms}) \mod (v_1, \ldots, v_{k-1})$.
From this equation, we get a similar one for the inverse of f:

$$f^{-1}(X) \equiv X - \frac{v_k}{\pi} X^{q^k} + (\text{higher order terms}) \mod (v_1, \ldots, v_{k-1}).$$

From here, it is easy to see that

$$\pi_F(X) = f^{-1}(\pi f(X)) \equiv \pi X + v_k(1-\pi^{q^k-1}) + (\text{higher order terms}) \mod (v_1, \dots v_{k-1}),$$

and thus that

$$\pi_F(X) \equiv v_k X^{q^k} + (\text{higher order terms}) \mod (\pi, v_1, \dots, v_{k-1}). \tag{2.32}$$

Now, let $F_{\underline{u}}$ be the $\mathcal{O}_K$-typical module over $\mathcal{O}_K[\![u_1, \dots, u_{h-1}]\!]$, which is the specialization of F with respect to the morphism

$$\mathcal{O}_K[\underline{v}] \to \mathcal{O}_K[\![u_1, \dots, u_{h-1}]\!],$$
$$v_i \mapsto \begin{cases} u_i, & \text{if } i < h \\ 1, & \text{if } i = h \\ 0, & \text{if } i > h. \end{cases}$$

It is a formal $\mathcal{O}_K$-module of height h, as is seen from 2.32. From equation 2.27 one also sees that its reduction $\overline{F}$ modulo the maximal ideal $(\pi, u_1, \dots, u_{h-1})$ is additive modulo degree q^h.
Therefore, we are in the situation of the last section, Section 2.4. We know from this section that the Lubin-Tate deformation functor associated to $\overline{F}$ is represented by $\mathcal{O}_K[\![\underline{u}]\!] = \mathcal{O}_K[\![u_1, \dots, u_{h-1}]\!]$.
By 2.32 we know that $F_{\underline{u}}$ has specializations $\overline{F}^i$ over $k[\varepsilon]$ of the form

$$\pi_{F^i}(X) = \varepsilon X^{q^i} + (\text{higher order terms})$$

for $i \in \{1, \dots, h-1\}$. The first order deformations $\overline{F}^i$ correspond to cocycles that form a basis for $H^2(\overline{F}, \mathbb{G}_a)_s$ by Theorem 2.4.1.
Thus, as in the proof of representability of Lubin-Tate space, the morphism $\mathcal{O}_K[\![\underline{u}]\!] \to \mathcal{O}_K[\![\underline{u}]\!]$ giving $F_{\underline{u}}$ as specialization of the universal deformation F_{univ} of $\overline{F}$, is actually an isomorphism, as it is an isomorphism on the tangent spaces. That is, the $\mathcal{O}_K$-typical module $F_{\underline{u}}$ is a universal deformation of $\overline{F}$. In particular, every deformation of $\overline{F}$ (over a complete Noetherian local $\mathcal{O}_K$-algebra) is $\star$-isomorphic to an $\mathcal{O}_K$-typical one.
The universal deformation $F_{\underline{u}}$ is nice to work with, but of course it can only be used when studying the deformations of an $\mathcal{O}_K$-typical module over k. By Proposition 2.3.13, after base-changing to $\overline{k}$, every height h formal $\mathcal{O}_K$-module over k becomes isomorphic to the $\mathcal{O}_K$-typical $\overline{F}$. For this reason, studying the deformation theory of an $\mathcal{O}_K$-typical module is, for most purposes, general enough.

Remark 2.5.2. *We already note one nice property of $F_{\underline{u}}$: its reduction $\overline{F}$ is normal, that is $\pi_{\overline{F}}(X) = X^{q^h}$. (By 2.27 we already know that $\overline{F}$ is additive up to degree $q^h - 1$.)*

Proof. Indeed, consider the specialization $F_{\underline{0}}$ of $F_{\underline{u}}$ obtained from the morphism $\mathcal{O}_K[\![\underline{u}]\!] \to \mathcal{O}_K$ with kernel $(\underline{u})$. By the "functional equation" its logarithm $g(X)$ satisfies $g(X) = X + \pi^{-1} g(X^{q^h})$, whence

$$g(X) = \sum_{i \geq 0} \pi^{-i} X^{q^{ih}}.$$

Now we prove by induction that

$$\pi_{F_{\underline{0}}}(X) \equiv X^{q^h} + \pi P(X) \mod \text{degree } n$$

for some $P(X) \in \mathcal{O}_K[\![X]\!]$ and where $n > q^h$. The case $n = q^h + 1$ follows from 2.32.
Assume that we have proven the claim for some $n > q^h$. Then write

$$\pi_{F_{\underline{0}}}(X) \equiv X^{q^h} + \pi P(X) + \alpha X^n \mod \text{degree } n+1$$

for some $\alpha \in \mathcal{O}_K$. We want to show $\alpha \in \pi\mathcal{O}_K$. Applying g to both sides of the congruence gives, modulo degree $n+1$,

$$\begin{aligned}
\pi g(X) &\equiv g\left(X^{q^h} + \pi P(X) + \alpha X^n\right) \\
&\equiv X^{q^h} + \pi P(X) + \alpha X^n + \sum_{i>0} \pi^{-i}\left(X^{q^h} + \pi P(X) + \alpha X^n\right)^{q^{ih}} \\
&\equiv X^{q^h} + \pi P(X) + \alpha X^n + \sum_{i>0} \pi^{-i}\left(X^{q^h} + \pi P(X)\right)^{q^{ih}}.
\end{aligned}$$

By Newton's binomium

$$x \equiv y \mod \pi^i \implies x^{q^h} \equiv y^{q^h} \mod \pi^{i+1}$$

for all $x, y \in \mathcal{O}_K[\![X]\!]$. Therefore, we find that

$$\left(X^{q^h} + \pi P(X)\right)^{q^{ih}} \equiv X^{q^{(i+1)h}} \mod \pi^{i+1}\mathcal{O}_K[\![X]\!].$$

This gives us, modulo degree $n+1$ and modulo $\pi\mathcal{O}_K[\![X]\!]$, that

$$\begin{aligned}
\pi g(X) &\equiv X^{qh} + \alpha X^n + \sum_{i>0} \pi^{-i} X^{q^{(i+1)h}} \\
&= \pi g(X) - \pi X + \alpha X^n.
\end{aligned}$$

It follows that $\alpha \in \pi\mathcal{O}_K$, which finishes the proof. ■

2.6 The Gross-Hopkins period morphism

In Theorem 2.4.1 we computed the dimension of $H^2(\widetilde{F}, \mathbb{G}_a)_s$ for a deformation $\widetilde{F}$ of a height h formal $\mathcal{O}_K$-module $\overline{F}/k$ over an arbitray complete Noetherian local $\mathcal{O}_K$-algebra R. This was not necessary in order to compute $\mathcal{M}_{\overline{F},0}$: to apply Schlessinger's deformation theorem one only needs to know the second symmetric cohomology $H^2(\overline{F}, \mathbb{G}_a)_s$ over the residue fields k_R. It was also not necessary for the next chapter of this thesis. However, the computation of the general second cohomology with values in $\mathbb{G}_a$ is very useful in the study of Lubin-Tate spaces as it is a necessary step in the construction of the Gross-Hopkins period morphism. In this section we will give a sketch of the construction of this morphism as done in [GH94]. We warn the reader that this section will be a very rough summary and therefore lack greatly in terms of details.

The utility of the more general computation of the symmetric 2-cohomology lies in the far-reaching consequences of Proposition 2.2.7 that

$$H^2(\widetilde{F}, \mathbb{G}_a)_s \cong \mathrm{Ext}(\widetilde{F}, \mathbb{G}_a).$$

Namely, setting $M \coloneqq \mathrm{Hom}\big(\mathrm{Ext}(\widetilde{F}, \mathbb{G}_a), R\big)$, one can define the $h-1$-dimensional $\mathcal{O}_K$-module

$$V(\widetilde{F}) \coloneqq \mathbb{G}_a \otimes M,$$

which is given by the natural structure of an $\mathcal{O}_K$-module object on the $S^{\geq 1}M$-adic completion of the R-algebra of symmetric powers $S^0M \oplus S^1M \oplus \cdots$. Then

$$\begin{aligned}
\mathrm{Ext}\big(\widetilde{F}, V(\widetilde{F})\big) &\cong H^2(\widetilde{F}, \mathbb{G}_a \otimes M)_s \\
&\cong H^2(\widetilde{F}, \mathbb{G}_a)_s \otimes_R M \\
&\cong \mathrm{Ext}(\widetilde{F}, \mathbb{G}_a) \otimes_R \mathrm{Hom}\big(\mathrm{Ext}(\widetilde{F}, \mathbb{G}_a), R\big) \\
&\cong \mathrm{End}\big(\mathrm{Ext}(\widetilde{F}, \mathbb{G}_a)\big).
\end{aligned}$$

Let

$$0 \to V(\widetilde{F}) \to E(\widetilde{F}) \to \widetilde{F} \to 0$$

be the extension corresponding to the identity of $\mathrm{Ext}(\widetilde{F}, \mathbb{G}_a)$. Then one has canonically that $\mathrm{Ext}(\widetilde{F}, V') \subset \mathrm{Hom}\big(V(\widetilde{F}), V'\big)$ for any additive formal $\mathcal{O}_K$-module V' over R. Therefore, any *additive* extension of $\widetilde{F}$

$$0 \to V' \to E' \to \widetilde{F} \to 0$$

corresponds to a morphism $g\colon V(\widetilde{F}) \to V'$, which induces a morphism $i\colon E(\widetilde{F}) \to E'$. Together, they are unique morphisms such that the following diagram commutes:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & V(\widetilde{F}) & \longrightarrow & E(\widetilde{F}) & \longrightarrow & \widetilde{F} & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle g} & & \big\downarrow{\scriptstyle i} & & \big\downarrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & V' & \longrightarrow & E' & \longrightarrow & \widetilde{F} & \longrightarrow & 0
\end{array}.$$

That is, $E(\widetilde{F})$ is the initial object of the category of additive extensions of $\widetilde{F}$. It is called the *universal additive extension of* $\widetilde{F}$ and also the *universal vector extension*.

There are many reasons $E(\widetilde{F})$ is interesting. One is that the free rank h R-module of invariant differentials $\omega\big(E(\widetilde{F})\big)$ has an interpretation as the *R-module of rigidified extensions of $\widetilde{F}$ by $\mathbb{G}_a$* $\mathrm{RigExt}(\widetilde{F}, \mathbb{G}_a)$, which is the set of isomorphism classes of extensions $0 \to \mathbb{G}_a \to E \to \widetilde{F} \to 0$ together with a splitting of the induced short exact sequence of Lie algebras. Equivalently, $\mathrm{RigExt}(\widetilde{F}, \mathbb{G}_a)$, and therefore $\omega\big(E(\widetilde{F})\big)$, can be seen, à la Proposition 2.2.7, as the set of $\star$-isomorphism classes of first order deformations $\widehat{\widetilde{F}}$ of $\widetilde{F}$ over $R[\varepsilon]$ together with an invariant differential ω of $\widehat{\widetilde{F}}$ that reduces to the canonical basis element $\omega_{\widetilde{F}}$ modulo ε.

One can make things even more universal by applying the above construction to $R = \mathcal{O}_K[\![u_1, \ldots, u_{h-1}]\!]$ and $\widetilde{F} = F_{\mathrm{univ}}$. In [GH94] $\overline{F}$ is $\mathcal{O}_K$-typical and we can take the good coordinates $F_{\mathrm{univ}} = F_{\underline{u}}$ of the previous section.

To construct the period morphism, Gross and Hopkins enter the world of rigid

analytic geometry: they show that the generic fibre of $\mathcal{M}_{\overline{F},0}$ can be seen as a "rigid analytic space" $\mathfrak{X} \coloneqq \mathcal{M}_{\overline{F},0}^{\mathrm{rig}}$ over K [GH94][§18]. Then they describe how one can view $\mathrm{RigExt}(F_{\underline{u}}, \mathbb{G}_a) = \omega\big(E(F_{\underline{u}})\big)$ and its dual $\mathrm{Lie}\, E(F_{\underline{u}})$ as "rigid vector bundles" over $\mathfrak{X}$.
A very important aspect of their paper is that $\mathfrak{X}$ and the considered vector bundles over it have natural actions of the group $\mathrm{Aut}(\overline{F})$ of automorphisms of F over k. We leave it to the reader to check that $\mathrm{Aut}(\overline{F})$ equals the unit group of the non-commutative $\mathcal{O}_K$-algebra

$$\mathrm{End}(\overline{F}) = \mathcal{O}_K[\Phi],$$

where $\Phi(X) = X^q$ is the q-Frobenius, satisfying $\Phi^h = \pi$ by Remark 2.5.2.
The actions of $\mathcal{O}_K[\Phi]^\times$ on $\mathfrak{X}$ and $\mathrm{Lie}\, E(F_{\underline{u}})$ are compatible, meaning that the vector bundle is "equivariant" (this is also the case for the other vector bundles that are considered in the paper). What does not hold for in general for the other vector bundles considered, is that $\mathrm{Lie}\, E(F_{\underline{u}})$ is "flat"[GH94][Prop. 22.4]. That is, there is an h-dimensional K-vector space M with an action of $\mathcal{O}_K[\Phi]^\times$ and an isomorphism

$$\mathrm{Lie}\, E(F_{\underline{u}}) \cong M \otimes_K \mathcal{O}_{\mathfrak{X}}$$

that respects the actions of $\mathcal{O}_K[\Phi]^\times$, letting it act diagonally on the left hand side. The sequence of Lie algebras induced by the universal additive extension

$$0 \to \mathrm{Lie}\, V(F_{\underline{u}}) \to \mathrm{Lie}\, E(F_{\underline{u}}) \to \mathrm{Lie}\, F_{\underline{u}} \to 0$$

is a short exact sequence of equivariant vector bundles over $\mathfrak{X}$ (i.e., all three vector bundles have actions of $\mathcal{O}_K[\Phi]^\times$ that are compatible with the action on $\mathfrak{X}$ and the above morphisms are equivariant for these actions). This yields an equivariant surjection

$$M \otimes_K \mathcal{O}_{\mathfrak{X}} \cong \mathrm{Lie}\, E(F_{\underline{u}}) \twoheadrightarrow \mathrm{Lie}\, F_{\underline{u}}.$$

Since $\mathrm{Lie}\, F_{\underline{u}}$ has rank 1, this induces, by the universal property of (the rigid analytic) projective space, a morphism

$$\pi_{GH} \colon \mathcal{M}_{\overline{F},0}^{\mathrm{rig}} = \mathfrak{X} \to \mathbb{P}(M) \cong \mathbb{P}^h$$

of rigid analytic spaces over K, called the *Gross-Hopkins period morphism*.
The $\mathcal{O}_K[\Phi]^\times$-action on M induces one on $\mathbb{P}(M)$ (coordinate-wise) and the period morphism is seen the be equivariant for the actions on both sides. Gross and Hopkins go on to prove it is étale, and surjective over $\overline{K}$ [GH94][Prop. 23.5]. This was actually proven already by Lafaille in 1978 [Laf79] (though in less generality, when K has characteristic 0).
In the paper of Gross and Hopkins the O_K-typicality of $F_{\underline{u}}$ is crucial at multiple points. Firstly, it allows them to give a very explicit isomorphism

$$\mathrm{RigExt}(F_{\underline{u}}, \mathbb{G}_a) \cong \frac{\{g(X) \in (\mathcal{O}_K[\![\underline{u}]\!] \otimes_{\mathcal{O}_K} K)[\![X]\!] \colon g(0) = 0, \delta g \text{ and } g' \text{ are integral}\}}{\{g(X) \in \mathcal{O}_K[\![\underline{u}]\!][\![X]\!] \colon g(0) = 0\}},$$

where δg is the coboundary of g as in Definition 2.2.6 and where we call a power series "integral" if its coefficients are in $\mathcal{O}_K[\![\underline{u}]\!]$ – this is [GH94][Prop. 8.5]. This description allows them to prove flatness of the equivariant vector bundle $\mathrm{Lie}\, E(F_{\underline{u}})$ dual to $\mathrm{RigExt}(F_{\underline{u}}, \mathbb{G}_a)$. It also enables them to give a very explicit formula for

the period morphism ([GH94][Eq. 23.6]), which ultimately can be yielded to prove its étaleness and its surjectivity over $\overline{K}$.
We would finally like to note that one can do the same constructions after base-change to the unramfied degree h extension of K. Then $\mathcal{O}_K[\Phi]^\times$ gets replaced by the automorphism group of $\overline{F}$ *over* $\mathbb{F}_{q^h}$, which is the group of units of the maximal order of D, the central division algebra over K of invariant $1/h$, see [Dri74][Prop. 1.7]. This is also done in [GH94]. From here on, one can get the whole action of $D^\times$, by replacing Lubin-Tate space by countably infinite many copies of itself, which is quite a standard thing to do, see [Ans21][p. 92 below]. Then we still have the period morphism, and it is then $D^\times$-equivariant. At last, we mention that this period morphism is one of the ingredients of Scholze's functor, which we mentioned in the introduction [Sch15].

Chapter 3

The higher level Lubin-Tate spaces

In the previous chapter we have seen that the Lubin-Tate deformation functor of a height $h < \infty$ formal $\mathcal{O}_K$-module $\overline{F}$ over k is represented by

$$\mathcal{M}_{\overline{F},0} \cong \operatorname{Spf} \mathcal{O}_K[\![u_1, \dots, u_{h-1}]\!].$$

One can thus think of $\mathcal{M}_{\overline{F},0}$ as the "open $h-1$-dimensional unit disk". In this chapter we will define the higher level Lubin-Tate spaces, that are not as simple anymore. In the end of the chapter, we show that "considering all levels at once", one gets a space which is of perfectoid nature in some sense. For this, we heavily follow Weinstein's paper [Wei10].

3.1 Drinfeld level structure

The higher level Lubin-Tate spaces are defined by requiring some structure on the π^n-torsion parts of the deformations. Let us introduce some notation. For any formal $\mathcal{O}_K$-module F over some $\mathcal{O}_K$-algebra R and any $n \geq 0$, we let $F[\pi^n]$ denote the functor from the category of complete adic R-algebras $\mathbf{CA}_R$ to the category of $\mathcal{O}_K$-modules that sends $S \in \mathbf{CA}_R$ to

$$F[\pi^n](S) \coloneqq \{x \in F(S) \colon \pi^n x = 0\}.$$

The following definition is due to Drinfeld [Dri74][§4B]:

Definition 3.1.1. *Let R be a complete local $\mathcal{O}_K$-algebra with the $\mathfrak{m}_R$-adic topology and F a 1-dimensional formal $\mathcal{O}_K$-module over R of height $h < \infty$. We define a* Drinfeld level π^n structure on F *to be a morphism of $\mathcal{O}_K$-modules*

$$\varphi \colon (\pi^{-n}\mathcal{O}_K/\mathcal{O}_K)^h \to F[\pi^n](R)$$

such that $\pi_F(X)$ is divisible by

$$\prod_{x \in (\pi^{-1}\mathcal{O}_K/\mathcal{O}_K)^h} (X - \varphi(x))$$

in $R[\![X]\!]$.
The images under such φ of the standard basis elements

$$(\pi^{-n}, 0, \dots, 0), (0, \pi^{-n}, 0, \dots, 0), \dots, (0, \dots, 0, \pi^{-n})$$

are called a Drinfeld basis of $F[\pi^n]$ over R.

Remark 3.1.2. *Notice that $x_1, \ldots, x_h$ is a Drinfeld basis for $F[\pi^n]$ over R if and only if $\pi^{n-1}x_1, ;\pi^{n-1}x_h$ is one for $F[\pi^n](R)$.*

Remark 3.1.3. *The word "basis" suggest perhaps linear independence between the different elements making up a Drinfeld basis, but this is not the case! In fact, if F is of height h and defined over k, then the $\pi_F^n(X)$ are equal to the product of $X^{q^{nh}}$ with a unit, and thus the only Drinfeld basis for $F[\pi^n](k)$ would be $0, \ldots, 0$.*

The following version of the Weierstraß preparation theorem will be useful in the study of Drinfeld level structures:

Proposition 3.1.4. *Let A be an I-adically complete ring. Suppose $n \geq 0$ and $f = \sum_{i \geq 0} a_i X^i \in A[\![X]\!]$ such that $a_n \in A^\times$ and $a_i \in I$ for all $i < n$. Then, for any $g \in A[\![X]\!]$ there is a $Q \in A[\![X]\!]$ and an $r \in A[X]$ of degree at most $n-1$ such that*

$$g = Qf + r.$$

Proof. Modulo I this statement is clear, because then f is the product of X^n and a unit of $A[\![X]\!]$. So now suppose r_1 (of degree $< n$) and Q_1 are lifts of this solution modulo I, that is, suppose they satisfy

$$g_1 \coloneqq g - (Q_1 f + r_1) \in IA[\![X]\!].$$

Again, on the $A[\![X]\!]$-module $IA[\![X]\!]/I^2A[\![X]\!]$ the action of f is that of multiplication by X^n and by a unit, so we find $r_2 \in IA[X]$ (of degree $< n$) and $Q_2 \in IA[\![X]\!]$ such that

$$g_2 \coloneqq g_1 - (Q_2 f + r_2) \in I^2 A[\![X]\!].$$

Going on like this, one finds the I-adically converging series

$$Q \coloneqq \sum_i Q_i \text{ and } r \coloneqq \sum_i r_i,$$

which indeed satisfy $g = Qf + r$ by I-adic completeness. ∎

Corollary 3.1.5. *If A is a complete local ring and $n \geq 0$, $f = \sum_i a_i X^i \in A[\![X]\!]$ such that $a_n \in A^\times$ and $a_i \in \mathfrak{m}_A$ for $i < n$, then there is a monic degree n polynomial $P(X) \in A[X]$ and a unit $u(X) \in A[\![X]\!]^\times$ such that*

$$f = uP.$$

Proof. Apply the preceding proposition to $g(X) = X^n$ to find a polynomial r of degree $< n$ and a power series Q such that $X^n - r = fQ$. Reducing modulo $\mathfrak{m}_A$ tells us Q is a unit, whence we are done. ∎

Remark 3.1.6. *From the above corollary it follows that, in the setting of the corollary (with* finite *height!), $F[\pi^n]$ is represented by $R[\![X]\!]/(P_n)$ and is finite free over R. In fact, from this one deduces the structure of a "co-$\mathcal{O}_K$-module object" on $R[\![X]\!]/(P_n)$, i.e. its spectrum is a (finite free) $\mathcal{O}_K$-module scheme over R. Thus, $F[\pi^n]$ can be viewed as a finite free $\mathcal{O}_K$-module scheme over R.*

The above proposition and its corollary tell us multiple things about the Drinfeld level structures. Firstly, it tells us something about the π-power torsion points:

Corollary 3.1.7. *If F is a formal $\mathcal{O}_K$-module of height $h < \infty$ over a complete local $\mathcal{O}_K$-algebra R, then for every $n \geq 0$, there is a (unique) monic polynomial $P_n \in R[X]$ of degree q^{nh} and a unit $u_n \in R[\![X]\!]^\times$ such that*

$$\pi_F^n = u_n P_n.$$

In particular, the roots of π_F^n are contained in an integral extension of R and if $\varphi : (\pi^{-n}\mathcal{O}_K/\mathcal{O}_K) \to F[\pi^n](R)$ is a Drinfeld level π^n structure, then

$$P_1(X) = \prod_{x \in (\pi^{-1}\mathcal{O}_K/\mathcal{O}_K)^h} (X - \varphi(x)).$$

Furthermore, the Weierstraß preparation theorem allows for an alternative definition of Drinfeld level structure that takes the formal $\mathcal{O}_K$-module structure more into account:

Corollary 3.1.8 ([Far05][5th Course, Cor. 1])**.** *Let F be a formal $\mathcal{O}_K$-module of height $h < \infty$ over a complete local $\mathcal{O}_K$-algebra R. Then a morphism $\varphi : (\pi^{-n}\mathcal{O}_K/\mathcal{O}_K)^h \to F[\pi^n](R)$ is a Drinfeld level π^n structure if and only if*

$$\prod_{x \in (\pi^{-1}\mathcal{O}_K/\mathcal{O}_K)^h} (X -_F \varphi(x))$$

divides π_F in $R[\![X]\!]$.

Proof. Consider the ring $A \coloneqq R[\![Y]\!]$, which is Y-adically complete. Set $f(X) = X - Y \in A[\![X]\!]$ and apply Proposition 3.1.4 to $g(X) = X -_F Y \in A[\![X]\!]$. We find $r \in A[X]$ of degree < 1 and $Q \in A[\![X]\!] = R[\![X, Y]\!]$ such that

$$X -_F Y = (X - Y)Q(X, Y) + r(Y).$$

If we plug in $Y = X$ in the above equation, we find that $r = 0$, and then, since $X -_F Y \equiv X - Y \mod \text{degree } 2$ it follows that Q is a unit.
Therefore, $X - Y$ and $X -_F Y$ differ by a unit, and thus so do

$$\prod_{x \in (\pi^{-1}\mathcal{O}_K/\mathcal{O}_K)^h} (X - \varphi(x)) \text{ and } \prod_{x \in (\pi^{-1}\mathcal{O}_K/\mathcal{O}_K)^h} (X -_F \varphi(x)).$$

■

Remark 3.1.9. *Finally, with a bit more effort, using the Weierstraß preparation theorem and its Corollary 3.1.8 one can give another alternative definition for Drinfeld level π^n level structure: namely, under the assumptions of Definition 3.1.1, the condition that the polynomial*

$$\prod_{x \in \pi^{-n}\mathcal{O}_K/\mathcal{O}_K} (X - \varphi(x)) \tag{3.1}$$

divides π_F^n is equivalent to the statement that φ is a Drinfeld level π^n structure. By Weierstraß preparation this condition is the same as the one that π_F^n is a unit times the product 3.1. The equivalence of this definition with ours is proven in [Far05][5th course, Prop. 1]. The proof there is less general ($\mathcal{O}_K = \mathbb{Z}_p$), but can be adjusted to the general case by replacing each p by either π or q (one only needs to know which replacement comes when, but this is clear once the proof is understood).
The equivalence of this definition shows that a Drinfeld basis for $F[\pi^n](R)$ in some sense can be called a "basis" after all: (when R is a domain) it shows all the roots of π^n are in the span of the Drinfeld basis.

The higher level Lubin-Tate spaces

Using the notion of Drinfeld level structure, we can define the higher level Lubin-Tate spaces associated to $\overline{F}$. A priori, we can only define them as functors.

Definition 3.1.10. *The* Lubin-Tate functor of level n associated to $\overline{F}$ *is defined to be*

$$\mathcal{M}_{\overline{F},n} : \mathcal{C} \to \mathbf{Set},$$
$$R \mapsto \left\{ \begin{array}{l} \textit{isomorphism classes of pairs } (\widetilde{F}, \varphi), \\ \textit{where } \widetilde{F} \textit{ is a deformation of } \overline{F} \textit{ and } \varphi \\ \textit{a Drinfeld level } \pi^n \textit{ structure on } \widetilde{F} \textit{ over } R \end{array} \right\},$$

where an isomorphism between two pairs $(\widetilde{F}, \varphi)$ *and* $(\widetilde{F}', \varphi')$*, as in the definition, is a* $\star$*-isomorphism* $f : \widetilde{F} \xrightarrow{\sim} \widetilde{F}'$ *such that the following diagram commutes*

$$\begin{array}{ccc} (\pi^{-n}\mathcal{O}_K/\mathcal{O}_K)^h & \xrightarrow{\varphi} & \widetilde{F}[\pi^n](R) \\ & \searrow^{\varphi'} & \downarrow f \\ & & \widetilde{F}'[\pi^n](R) \end{array} . \tag{3.2}$$

For all $m, n \geq 0$ we have natural transformations

$$\mathcal{M}_{\overline{F},n+m} \to \mathcal{M}_{\overline{F},n}$$

given on points R by sending (the class of) a pair $(\widetilde{F}, \varphi)$ to (the class of) $(\widetilde{F}, \varphi \circ \iota)$ (or, if $n = 0$, simply to the class of $\widetilde{F}$), where ι is the inclusion

$$\pi^{-n}\mathcal{O}_K/\mathcal{O}_K \hookrightarrow \pi^{-m-n}\mathcal{O}_K/\mathcal{O}_K.$$

This way, we get what is called the *Lubin-Tate tower (associated to* $\overline{F}$*)*

$$\cdots \to \mathcal{M}_{\overline{F},2} \to \mathcal{M}_{\overline{F},1} \to \mathcal{M}_{\overline{F},0}. \tag{3.3}$$

In this chapter we will see that these functors are all formal schemes and that the morphisms described above are finite flat. We will then discuss the proof of [Wei10] that the "projective limit" of the above diagram is perfectoid (in some sense). However, we will first illustrate the height 1 case, in which everything is relatively simple.

3.2 The height 1 case and its perfectoid nature

Suppose $\overline{F}$ is a 1-dimensional formal $\mathcal{O}_K$-module over k of height 1. Then we know its Lubin-Tate space is just the point

$$\operatorname{Spf} \mathcal{O}_K.$$

This means that lifts of $\overline{F}$ are unique up to $\star$-isomorphism. Let F be one such lift over $\mathcal{O}_K$. Then F is a universal deformation and for any complete Noetherian local $\mathcal{O}_K$-algebra R we can, by specialization, view F as a representative of the unique $\star$-isomorphism class of deformations of $\overline{F}$ over R. It turns out that the Drinfeld bases for $F[\pi^n]$ over R correspond to roots of $\Phi_n(X) \coloneqq \pi_F^n(X)/\pi_F^{n-1}(X)$.

Proposition 3.2.1 ([Wei10][Lemma 2.3.1])**.** *Let R be a complete Noetherian local $\mathcal{O}_K$-algebra. Then an element $x \in F[\pi^n](R)$ is a Drinfeld basis if and only if it is a root of $\Phi_n(X)$.*

Proof. We can reduce to the case $n = 1$, because of Remark 3.1.2 and the fact that $\Phi_n(X) = \Phi_1\left(\pi_F^{n-1}(X)\right)$.
By definition, x is a Drinfeld basis for $F[\pi](R)$ if and only if $T\prod_{a\in k^\times}(X - a_F(X))$ divides $\pi_F(X)$. So if x is a Drinfeld basis for $F[\pi](R)$, then clearly $\Phi_1(x) = 0$.
Conversely, if x is a zero of Φ_1, so is $a_F(x)$, for any $a \in \mathcal{O}_K^\times$ (this implies x is a Drinfeld basis for $F[\pi](R)$): indeed, we can write

$$a_F(X) = aX \cdot \eta(X),\ \eta \in A[\![X]\!]^\times,$$

from which we get

$$\Phi_1\left(a_F(X)\right) = \frac{\pi_F\left(a_F(X)\right)}{a_F(X)} = \frac{a\cdot\pi_F(X)}{aX}\cdot\underbrace{\frac{\eta\left(\pi_F(X)\right)}{\eta(X)}}_{\text{unit}}.$$

Therefore, if x is a root of $\Phi_1(X) = \pi_F(X)/X$, so is $a_F(x)$. ∎

We denote by K_n the field extension of K obtained by adjoining the roots of π_F^n (or Φ_n) to K.[1]

Remark 3.2.2. *We note that the extensions K_n/K are all totally ramified. When $\pi_F(X) = \pi X + X^q$ this is clear, because the Φ_n will be Eisenstein polynomials. In the general case, we know by Proposition 2.3.12 that there is a change of coordinates over the maximal unramified extension $\mathcal{O}_{K^{unr}}$ such that $F \mod \pi$ becomes normal. Therefore, the π-power torsion points of F in the original coordinates are obtained by evaluating the roots of $\pi X + X^q$ and its iterates in an invertible power series with coefficients in $\mathcal{O}_{K^{unr}}$; such a change of coordinates is an isometry on $\{x \in \overline{K} \colon |x| < 1\}$ and thus we have shown that in general K_n/K is a totally ramified extension, and we also see that the degree of this extension must be* $\deg \Phi_n$.

From the above proposition it follows that the level n Lubin-Tate functor $\mathcal{M}_{\overline{F},n}$ is the formal scheme

$$\operatorname{Spf} \mathcal{O}_{K_n}.$$

By considering the compositum $K_\infty \coloneqq \bigcup_n K_n$ and its π-adic completion $\hat{K}_\infty$, we get an isomorphism of functors on $\mathcal{C}$

$$\mathcal{M}_{\overline{F},\infty} \coloneqq \varprojlim_n \mathcal{M}_{\overline{F},n} \cong \operatorname{Hom}(\mathcal{O}_{K_\infty}, -) = \operatorname{Hom}(\mathcal{O}_{\hat{K}_\infty}, -),$$

which sends a complete Noetherian local $\mathcal{O}_K$-algebra R to

$$\left\{\begin{array}{l}\text{isomorphism classes of pairs } (\widetilde{F}, \varphi),\\ \text{where } \widetilde{F} \text{ is a deformation of } \overline{F} \text{ and } \varphi\\ \text{a ``Drinfeld level } \pi^\infty \text{ structure on } \widetilde{F} \text{ over } R\text{''}\end{array}\right\},$$

[1] Note that we just need to add one root of Φ_n to K to obtain K_n, because applying the a_F then gives us the other roots.

where a *level π^∞ level structure φ on $\widetilde{F}$ over R* is a morphism of $\mathcal{O}_K$-modules

$$\varphi\colon K/\mathcal{O}_K \to \widetilde{F}[\pi^\infty](R) \coloneqq \varinjlim_n \widetilde{F}[\pi^n](R)$$

satisfying the same divisibility condition as in Definition 3.1.1 and where an isomorphism is defined similarly as in the definition of the finite levels. One has the following observations:

Proposition 3.2.3 ([Wei10][Lemma 2.3.2])**.** *The q-Frobenius is surjective on $\mathcal{O}_{K_\infty}/\pi = \mathcal{O}_{\hat{K}_\infty}/\pi$.*

Proof. Let $x_1, x_2, \dots$ be compatible roots of the $\Phi_1, \Phi_2, \dots$, respectively, that is, $\pi_F(x_{i+1}) = x_i$. Since $\pi_F(X) \equiv v(X^q) \mod \pi$ is a power series in X^q, every x_i has q-th root modulo π. The x_i generate the K_i over K, and therefore every element in $\mathcal{O}_K/\pi$ has a q-th root. ∎

Proposition 3.2.4. *The q-Frobenius $\mathcal{O}_{\hat{K}_\infty}/p \to \mathcal{O}_{\hat{K}_\infty}/p$ is surjective.*

Proof. Let $v \in \mathcal{O}_K[\![X]\!]$ and the $x_i \in \mathcal{O}_{\hat{K}_\infty}$ be as in the proof of the previous proposition. Then, since $\pi_F(X) \equiv (X^q) \equiv v(X)^q \mod \pi$ it can be seen that the sequence defined by $y_i \coloneqq v^{i-1}(x_i)$ for $i \geq 1$ satisfies $y_{i+1}^q \equiv y_i \mod \pi$ for all i. Since $|\pi| < |y_1| = |x_i|$ by Remark 3.2.2 it follows that

$$|y_i| = |\pi|^{1/q^i(q-1)}. \tag{3.4}$$

By the classical "$x \equiv y \mod \pi^i \implies x^p \equiv y^p \mod \pi^{i+1}$" we can see that the limits

$$\varpi_i \coloneqq \lim_{j\to\infty} y_{i+j}^{q^j}$$

exists. Clearly $|\varpi_i| = |y_i|$ and $\varpi_{i+j}^{q^j} = \varpi_i$ for all i and j.
The completion $\widetilde{\mathcal{O}}$ of $\mathcal{O}_K[\varpi_1, \varpi_2, \dots]$ inside $\mathcal{O}_{\hat{K}_\infty}$ is actually $\mathcal{O}_{\hat{K}_\infty}$ itself. Indeed, modulo π we have

$$\varpi_i \equiv \lim_{j\to\infty} v^{i+j-1}(x_{i+j})^{q^j} \equiv v^{i-1}(x_i) = y_i.$$

The x_i generate $\mathcal{O}_{\hat{K}_\infty}$ topologically over $\mathcal{O}_K$, therefore, so do the y_i, and thus the natural map $\widetilde{\mathcal{O}}/\pi \to \mathcal{O}_{\hat{K}_\infty}/\pi$ is an isomorphism. Since both rings are π-adically complete, a standard approximation argument lets us conclude that $\widetilde{\mathcal{O}} = \mathcal{O}_{\hat{K}_\infty}$.
Now, every element in $\widetilde{\mathcal{O}} = \mathcal{O}_{\hat{K}_\infty}$ can be expressed as a possibly infinite linear combination of monomials in the ϖ_i with coefficients in $\mathcal{O}_K$, whence Frobenius is surjective on $\mathcal{O}_{\hat{K}_\infty}/p$. ∎

Corollary 3.2.5. *If the characteristic of K is p, then*

$$\mathbb{F}_q[\![t^{1/q^\infty}]\!] = k[\![t^{1/q^\infty}]\!] \cong \mathcal{O}_{\hat{K}_\infty}$$

and thus

$$\hat{K}_\infty \cong \mathbb{F}_q(\!(t^{1/q^\infty})\!).$$

Proof. Indeed, the isomorphism can be defined by sending t^{1/q^i} to the ϖ_{i+1} as defined in the proof of Proposition 3.2.4. From the proof of the same proposition, we know this map must be surjective. It is also injective: giving t the same valuation as its image via this map, we can define an absolute value on $\mathbb{F}_q(\!(t^{1/q^\infty})\!)$ and the map becomes an isometry. ■

The field $\mathbb{F}_q(\!(t^{1/q^\infty})\!)$ is a complete non-Archimedean non-discretely valued field of characteristic p with that is perfect. It is one of the standard examples of a "perfectoid field":

Definition 3.2.6. *A* perfectoid field L *is a complete non-Archimedean field (of residue characteristic* p*) with a non-discrete valuation, such that Frobenius* $\mathcal{O}_L/p \to \mathcal{O}_L/p$ *is surjective.*

From the last proposition we have the following, also when K has characteristic zero:

Corollary 3.2.7. *The field* $\hat{K}_\infty$ *is perfectoid.*

The corollary states that the "generic fiber" of $\mathcal{M}_{\overline{F},\infty}$ is perfectoid. In the general case, where $\overline{F}$ has arbitrary finite height, the same is true, although in that case one needs a more general definition of "perfectoid".

When K has characteristic zero, we know it is perfectoid, but we do not have a description of $\hat{K}_\infty$ as in Corollary 3.2.5. We can come close, however, and give such a description for the "tilt" of $\hat{K}_\infty$. We briefly sketch the construction of the tilt of a perfectoid field L:

1) We define the ring

$$\mathcal{O}_{L^\flat} \coloneqq \varprojlim_{x \mapsto x^p} \mathcal{O}_L/p = \{(x_0, x_1, x_2, \dots) \in (\mathcal{O}_L/p)^{\mathbb{N}} \colon x_{i+1}^p = x_i\}.$$

2) The we define the multiplicative map $\sharp \colon \mathcal{O}_{L^\flat} \to \mathcal{O}_L$ by

$$x = (x_0, x_1, \dots) \mapsto x^\sharp \coloneqq \lim_{i \to \infty} x_i^{p^i},$$

we note that this limit always exists.

3) The map $\sharp$ allows us to define a non-Archimedean absolute value on $\mathcal{O}_{L^\flat}$ by $|x|_\flat \coloneqq |x^\sharp|$ (check this).

4) Then we define $L^\flat$ to be the field of fractions of $\mathcal{O}_{L^\flat}$ and we extend the non-Archimedean absolute value just defined to $L^\flat$; it turns out the ring of integers of $L^\flat$ is exactly $\varprojlim_{x \mapsto x^p} \mathcal{O}_L/p$, so there is no conflict of notation.

5) The field $L^\flat$ is perfectoid and of characteristic p; it is called the *tilt of* L.

Remark 3.2.8. *The tilt can also be constructed by setting*

$$\mathcal{O}_L^\flat \coloneqq \varprojlim_{x \mapsto x^p} \mathcal{O}_L/\pi'$$

for some topologically nilpotent non-zero $\pi'|p$. In fact, even using $\pi' = 0$ is allowed, even when $p \neq 0$, but then one needs to do some work to put a ring structure on

$$\varprojlim_{x \mapsto x^p} \mathcal{O}_L,$$

which is a priori just a set.

For more details, see [SW20]. A perfectoid field and its tilt share many similarities. For one thing, when L has characteristic p, it is equal to its tilt. More subtle similarities, which are relevant in the mixed characteristic case, are expressed in the tilting equivalence:

Theorem 3.2.9. *Let L be a perfectoid field.*

a) If M/L is a finite field extension, then M is perfectoid.

b) The functor $M \mapsto M^\flat$ is an equivalence of categories between finite extensions of L and finite extensions of $L^\flat$. Therefore, the absolute Galois groups of L and $L^\flat$ are isomorphic.

Proof. This is [SW20][Thm. 7.3.1], although it is originally due to Fontaine and Wintenberger. We do not use this theorem anywhere. We just state it in order to illustrate the importance of the tilt. ■

Proposition 3.2.10 ([Wei10][Prop. 2.3.3])**.** *The tilt $\hat{K}_\infty^\flat$ of $\hat{K}_\infty$ is isomorphic to $k(\!(t^{1/q^\infty})\!)$ and $\mathcal{O}_{\hat{K}_\infty^\flat} \cong k[\![t^{1/q^\infty}]\!]$.*

Proof. The projection on the first coordinate

$$\mathcal{O}_{\hat{K}_\infty^\flat} \to \mathcal{O}_{\hat{K}_\infty}/p$$

is surjective by Proposition 3.2.4 and thus $\hat{K}_\infty^\flat$ has residue field k. There is the Teichmüller multiplicative section $k \to \mathcal{O}_{\hat{K}_\infty^\flat}$, which is a ring morphism because $\hat{K}_\infty^\flat$ is of characteristic p.
Let

$$\varpi^\flat \coloneqq (\varpi_1, \dots, \varpi_2, \dots, \varpi_3, \dots) \in \mathcal{O}_{\hat{K}_\infty^\flat}$$

with the ϖ_i from Proposition 3.2.4. One has

$$(\varpi^\flat)^{1/q^i} = (\varpi_{i+1}, \dots, \varpi_{i+2,\dots})$$

and

$$\left((\varpi^\flat)^{1/q^i}\right)^\sharp = \varpi_{i+1}. \tag{3.5}$$

Now define

$$\varphi \colon k[\![t^{1/q^\infty}]\!] \to \mathcal{O}_{\hat{K}_\infty^\flat}, t \mapsto \varpi^\flat.$$

Let us define an absolute value on $k[\![t^{1/q^\infty}]\!]$ by setting $|t| \coloneqq |\varpi^\flat|_\flat = |\varpi_1|$. This makes φ into an isometry. As in Proposition 3.2.4 we see that the images of $k[\![t^{1/q^\infty}]\!]$ and $\mathcal{O}_{\hat{K}_\infty^\flat}$ under their respective absolute values are the same sets. Since both have the same residue field, we conclude by completeness that φ is an isomorphism. ■

Note that Corollary 3.2.5 is a also a corollary of the above proposition.

Remark 3.2.11. *The field K_∞ is important in local class field theory: $K^{ab} = K_\infty \cdot K^{unr}$, see for example [Mil20][I, Thm. 4.8]. This was first discovered by Lubin and Tate [LT65].*

3.3 The finite level Lubin-Tate spaces in the general case

We now study the other cases, so $\overline{F}$ is a formal $\mathcal{O}_K$-module over k of height $1 < h < \infty$. We write $A_0 \coloneqq \mathcal{O}_K[\![u_1, \dots, u_{h-1}]\!]$ for the $\mathcal{O}_K$-algebra representing $\mathcal{M}_{\overline{F},0}$.
Before we can define the infinite level Lubin-Tate space and prove its "generic fiber" is perfectoid we need to study the finite level Lubin-Tate spaces in some detail. Like the level structures themselves, these were first defined by Drinfeld in [Dri74][§4B]. All results in this section are due to him.

Proposition 3.3.1 ([Dri74][Prop. 4.3], [Far05][5th Course, Prop. 2])**.** *The level n Lubin-Tate functor $\mathcal{M}_{\overline{F},n}$ associated to $\overline{F}$ is represented by a complete local A_0-algebra A_n that is finite over A_0.*

Proof. If F_{univ} is the universal deformation of $\overline{F}$ (so a deformation over A_0), then consider the functor from $\mathcal{C}$ to **Set**

$$R \mapsto \left\{ \begin{array}{l} \text{isomorphism classes of pairs } (\widetilde{F}, \varphi), \\ \text{where } \widetilde{F} \text{ is a deformation of } \overline{F} \text{ and } \varphi \\ \text{an } \mathcal{O}_K\text{-linear morphism } (\pi^{-n}\mathcal{O}_K/\mathcal{O}_K)^h \to \widetilde{F}[\pi^n](R) \end{array} \right\},$$

where an isomorphism between pairs $(\widetilde{F}, \varphi), (\widetilde{F}', \varphi')$ is defined as a $\star$-isomorphism $f\colon \widetilde{F} \to \widetilde{F}'$ such that the diagram 3.2 commutes. This functor is

$$(F_{\text{univ}}[\pi^n])^h = F_{\text{univ}}[\pi^n] \times \cdots \times F_{\text{univ}}[\pi^n]$$

and is represented by the complete local a_0-algebra

$$B_n \coloneqq A_0[\![T_1, \dots, T_h]\!] / \left(\pi^n_{F_{\text{univ}}}(T_1), \dots, \pi^n_{F_{\text{univ}}}(T_h)\right).$$

The level n Lubin-Tate functor $\mathcal{M}_{\overline{F},n}$ is a subfunctor of it. For a complete Noetherian local $\mathcal{O}_K$-algebra R, an element $f \in \text{Hom}(B_n, R)$ represents the class of the pair $(f_*F_{\text{univ}}, \varphi_f)$, where φ_f is the morphism $(\pi^{-n}\mathcal{O}_K/\mathcal{O}_K)^h \to F_{\text{univ}}[\pi^n](R)$

$$(a_1, \dots, a_h) \mapsto (a_1)_{F_{\text{univ}}}(f(T_1)) +_{F_{\text{univ}}} \cdots +_{F_{\text{univ}}} (a_h)_{F_{\text{univ}}}(f(T_h)).$$

Thus $f \in \text{Hom}(B_n, R)$ is in $\mathcal{M}_{\overline{F},n}$ if and only if $f(T_1), \dots, f(T_h)$ is a Drinfeld basis for $f_*F_{\text{univ}}[\pi^n](R)$.
Set

$$\Psi(U) \coloneqq \prod_{(a_1,\dots,a_h) \in \pi^{-1}\mathcal{O}_K/\mathcal{O}_K} \Big(U - \big((a_1)_{F_{\text{univ}}}(T_1) +_{F_{\text{univ}}} \cdots +_{F_{\text{univ}}} (a_h)_{F_{\text{univ}}}(T_h)\big)\Big).$$

By Weierstraß preparation 3.1.4 write

$$\pi_{F_{\text{univ}}}(U) = \Psi(U)Q(U) + r(U), \tag{3.6}$$

where $Q(U) \in B_n[\![U]\!]$ and $r \in B_n[U]$ is of degree $< q^h$.
We see that $f \in \text{Hom}(B_n, R)$ lies in $\mathcal{M}_{\overline{F},n}$ if and only if $f_*r(U) = 0$. Thus, writing $r(U) = \sum_{i=0}^{q^h-1} b_i U^i$, we find that the complete local Noetherian

$$A_n \coloneqq B_n/(b_0, \dots, b_{q^h-1})$$

represents the functor $\mathcal{M}_{\overline{F},n}$. Since B_n is already finite over A_0, we see that so is A_n. ∎

Recall that a regular local ring is a Noetherian local ring such that the minimal number of generators of its maximal ideal equals its Krull dimension. These type of rings have many nice properties. They are domains for example[Sta22, Tag 00NP].

Proposition 3.3.2. *A_n is a regular local ring, and thus in particular a domain.*

Proof. First of all, notice that A_0 injects into A_n: consider the extension $\widetilde{A}_0$ of A_0 obtained by adjoining all roots of $\pi^n_{F_{\text{univ}}}$. We pick roots $x_1, \dots, x_h \in \widetilde{A}_0$ of π_{univ^n} such that the $\pi^{n-1}_{F_{\text{univ}}}(x_i)$ span $F_{\text{univ}}[\pi](\widetilde{A}_0)$. This can be done, because $\pi^n_{F_{\text{univ}}}$ does not have repeated roots, which can be checked for example by reducing to the normal case as was done in Remark 3.2.2. This means that $x_1, \dots, x_h$ is a Drinfeld basis for $F_{\text{univ}}[\pi](\widetilde{A}_0)$. Therefore, we get an induced morphism $A_n \to \widetilde{A}_0$ by the universal property of A_n and the composition

$$A_0 \to A_n \to \widetilde{A}_0$$

is the inclusion $A_0 \subset \widetilde{A}_0$, because the formal $\mathcal{O}_K$-module on $\widetilde{A}_0$ is simply F_{univ}. It follows that A_0 injects into A_n.
Since A_n is finite over A_0 it is an integral extension of A_0. By the going-up and going-down theorems, it follows that A_n has the same Krull dimension as A_0. The Krull dimension of $A_0 = \mathcal{O}_K[\![u_1, \dots, u_{h-1}]\!]$ is $1 + (h-1)$, so $\dim A_n = h$.
The universal π^n level structure on F_{univ} corresponds to the universal Drinfeld basis for $F_{\text{univ}}[\pi^n](A_n)$, which is, using the construction of A_n as in the previous proposition, given by the images in A_n of the $T_1, \dots, T_h \in B_n$, which we will denote by the same.
We claim that $(T_1, \dots, T_h)$ is the maximal ideal of A_n. This claim would finish the proof, because the number of generators of the maximal ideal is at least the Krull dimension.
So let us prove the claim. Let $\rho\colon A_n \to A_n/(T_1, \dots, T_h)$ be the reduction morphism. The Drinfeld level π^n structure condition tells us that

$$U^{q^h} | \pi_{\rho_* F_{\text{univ}}}(U) = \rho_*(\pi_{F_{\text{univ}}})(U).$$

Therefore, the coefficients of the U^i with $i < q^h$ in $\pi_{F_{\text{univ}}}$ are all in $(T_1, \dots, T_h)$. By Proposition 2.4.4 it follows that $(\pi, u_1, \dots, u_{h-1})$ is in the kernel of the morphism $\mathcal{O}_K[\![u_1, \dots, u_h]\!] = A_0 \to A_n/(T_1, \dots, T_h)$. Thus, $A_n/(T_1, \dots, T_h)$ is a finite $k = A_0/(\pi, u_1, \dots, u_{h-1})$-algebra. The proof of the previous proposition shows that A_n is generated by the $T_1, \dots, T_h$ over A_0. Therefore, $A_n/(T_1, \dots, T_h)$ is actually equal to k and it follows that $(T_1, \dots, T_h)$ is maximal. ∎

Let us write $X_1^{(n)}, \dots, X_h^{(n)}$ for the universal Drinfeld basis for $F_{\text{univ}}[\pi^n](A_n)$ obtained from the $T_1, \dots, T_h \in B_n$ as in the proof of the above proposition. The proof shows that A_n is regular with parameters $X_1^{(n)}, \dots, X_h^{(n)}$.
Recall the morphisms making up the Lubin-Tate tower 3.3. They correspond to morphisms

$$A_i \to A_{i+j}, X_\ell^{(i)} \mapsto \pi^j_{F_{\text{univ}}}\left(X_\ell^{(i+j)}\right). \tag{3.7}$$

The previous proposition implies the following facts about these morphisms:

Corollary 3.3.3. *A_i injects into A_j for $j \geq i \geq 0$.*

Proof. A_j is an integral extension of $A_i/\mathfrak{p}$, where $\mathfrak{p}$ is the kernel of $A_i \to A_j$, which is a prime ideal of A_i. From $\dim A_i/\mathfrak{p} = \dim A_j = h = \dim A_i$ it follows that $\mathfrak{p} = 0$. ∎

Corollary 3.3.4. *A_j is flat over A_i for $j \geq i \geq 0$.*

Proof. This follows from the "miracle flatness theorem" [Sta22, Tag 00R4]. ∎

3.4 Universal covers and determinants

We need two more ingredients before we study the infinite level Lubin-Tate spaces in generality. The first we will discuss is the so-called "universal cover" of a formal $\mathcal{O}_K$-module. After that we will briefly mention some facts about determinants of formal $\mathcal{O}_K$-modules, the most important of which we do not prove. In this whole section we follow [Wei10].

The universal cover

Let us denote by $\overline{F}$ a height $h < \infty$ formal $\mathcal{O}_K$-module over k. It is a functor from the category of complete adic k-algebras $\mathbf{CA}_k$ to the category of $\mathcal{O}_K$-modules, $R \mapsto \overline{F}(R)$. We can use it to construct the *universal cover of* $\overline{F}$, denoted by $\mathrm{UC}(\overline{F})$, which is the functor from $\mathbf{CA}_k$ to the category of K-vector spaces $\mathbf{Vect}_K$ defined by

$$\mathrm{UC}(\overline{F})(R) \coloneqq \varprojlim_{x \mapsto \pi x} \overline{F}(R),$$

which is a K-vector space since it is an $\mathcal{O}_K$-module on which the endomorphism defined by π has the following inverse

$$(x_1, x_2, x_3, \dots) \mapsto (x_2, x_3, \dots).$$

Proposition 3.4.1 ([Wei12][Prop. 3.1.2])**.** *The functor* $\mathrm{UC}(\overline{F})$ *is represented by* $k[\![X^{1/q^\infty}]\!]$.

Proof. The endomorphism $\pi_{\overline{F}}$, being of height h, can be written as

$$u \circ \Phi^h = \Phi^h \circ u,$$

where $\Phi(X) = X^q$ is the q-Frobenius and u is an automorphism of $\overline{F}$ over k. We use this construct the isomorphism of $\mathcal{O}_K$-module valued functors

$$\begin{aligned} \mathrm{UC}(\overline{F}) &\to \varprojlim_{x \mapsto \Phi^h(x)} \overline{F}, \\ \mathrm{UC}(\overline{F})(R) &\to \varprojlim_{x \mapsto \Phi^h(x)} \overline{F}(R), \\ (x_1, x_2, x_3, \dots) &\mapsto \left(u(x_1), u^2(x_2), u^3(x_3), \dots\right), \end{aligned}$$

which is clearly well-defined and $\mathcal{O}_K$-linear, and which is an isomorphism because, on the level of points, the inverse is given by

$$(y_1, y_2, y_3, \dots) \mapsto \left(u^{-1}(y_1), u^{-2}(y_2), u^{-3}(y_3), \dots\right).$$

The q^h-th power Frobenius morphism $\Phi^h : \overline{F} \to \overline{F}$ corresponds to the endomorphism

$$\varphi_h \colon k[\![X]\!] \to k[\![X]\!], X \mapsto X^{q^h}.$$

Therefore,

$$\begin{aligned}\varprojlim_{-\circ\varphi_n} \operatorname{Hom}(k[\![X]\!], -) &\cong \operatorname{Hom}\left(\varinjlim_{\varphi_n} k[\![X]\!], -\right) \\ &= \operatorname{Hom}\left(k[\![X]\!][X^{1/q^\infty}], -\right) \\ &\cong \operatorname{Hom}\left(k[\![X^{1/q^\infty}]\!], -\right),\end{aligned}$$

where by abuse of notation the second and third homsets should be read as the sets of continuous morphisms of k-algebras, rather than homsets in $\mathbf{CA}_k$ (since $k[\![X]\!][X^{1/q^\infty}] \notin \mathbf{CA}_k$). This concludes the proof. ■

Suppose now that $\widetilde{F}$ is a lift of $\overline{F}$ to a complete Noetherian local $\mathcal{O}_K$-algebra R. We define its universal cover, a functor from the category of complete adic R-algebras to $\mathbf{Vect}_K$ this time, by the same formula

$$\mathrm{UC}(\widetilde{F}) \coloneqq \varprojlim_{x \mapsto \pi x} \widetilde{F}.$$

Proposition 3.4.2 ([Wei10][Prop. 2.4.2])**.** *Suppose S is a complete I-adic R-algebra with $\mathfrak{m}_R \subset I$. Then the reduction morphism*

$$\mathrm{UC}(\widetilde{F})(S) \to \mathrm{UC}(\widetilde{F})(S/I) = \mathrm{UC}(\overline{F})(S/I)$$

is an isomorphism. In this sense, $\mathrm{UC}(\widetilde{F})$ restricted to the category of complete adic R-algebras with an ideal of definition containing $\mathfrak{m}_R$ only depends on $\overline{F}$.

Proof. The reduction morphism is injective: for an element $(x_0, x_1, \dots) \in \mathrm{UC}(\widetilde{F})(S)$ in the kernel one has $x_{i+j} \in I$ for all i, j, whence, for every i, $x_i \in \bigcap_j \pi^j_{\widetilde{F}}(I) \subset \bigcap_j I^{j+1} = 0$.
To see surjectivity, let $(x_0 + I, x_1 + I, \dots) \in \mathrm{UC}(\widetilde{F})(S/I)$. We will apply by a Teichmüller lift kind of procedure. Since $\mathfrak{m}_R \subset I$ one has for every $\ell \geq 1$

$$x \equiv y \mod I^\ell \implies \pi_{\widetilde{F}}(x) \equiv \pi_{\widetilde{F}}(y) \mod I^{\ell+1}.$$

Therefore

$$\left(\lim_{j \to \infty} \pi^j_{\widetilde{F}}(x_{i+j})\right)_{i \geq 0}$$

is an element of $\mathrm{UC}(\widetilde{F})(S)$. It maps to $(x_0 + I, x_1 + I, \dots) \in \mathrm{UC}(\widetilde{F})(S/I)$. ■

Proposition 3.4.3 ([Wei10][Prop. 2.4.2])**.** *The universal cover $\mathrm{UC}(\widetilde{F})$ restricted to the category of complete adic R-algebras with an ideal of definition containing $\mathfrak{m}_R$ is representable by $R[\![X^{1/q^\infty}]\!]$.*

Proof. For S in this category write $\mathrm{TopNil}(S)$ for the set of topologically nilpotent elements of S. As a set, we know that $\widetilde{F}(S) = \mathrm{TopNil}(S)$. Notice that the morphism of multiplicative monoids induced by the reduction

$$\varprojlim_{x \mapsto x^{q^h}} \mathrm{TopNil}(S) \to \varprojlim_{x \mapsto x^{q^h}} \mathrm{TopNil}(S/\mathfrak{m}_R)$$

is an isomorphism by a standard Teichmüller lift procedure as done before. This allows us to conclude

$$\begin{aligned}\mathrm{UC}(\widetilde{F})(S) \overset{\text{Prop. 3.4.2}}{\cong} \mathrm{UC}(\widetilde{F})(S/\mathfrak{m}_R) &= \mathrm{UC}(\overline{F})(S/\mathfrak{m}_R) \\ &\overset{\text{Prop. 3.4.1}}{\cong} \varprojlim_{x\mapsto x^{q^h}} \mathrm{TopNil}(S/\mathfrak{m}_R) \\ &\cong \varprojlim_{x\mapsto x^{q^h}} \mathrm{TopNil}(S) \\ &\cong \mathrm{Hom}(R[\![X^{1/q^\infty}]\!], S).\end{aligned}$$

■

Remark 3.4.4. *The universal cover* $\mathrm{UC}(\widetilde{F})(S)$ *is related to the* Tate module $T(\widetilde{F})(S)$ *which is the sub-*$\mathcal{O}_K$*-module of* $\mathrm{UC}(\widetilde{F})(S)$ *defined by*

$$T(\widetilde{F})(S) \coloneqq \varprojlim_{n} \widetilde{F}[\pi^n](S).$$

When the topology on S *is discrete, every element of* $F(S)$ *is* π^n*-torsion for some* n*, and one obtains that*

$$\mathrm{UC}(\widetilde{F})(S) = T(\widetilde{F})(S) \otimes_{\mathcal{O}_K} K.$$

Determinants

We will soon see that the universal covers play an important role in the studying the strucure of the infinite level Lubin-Tate spaces. Namely, universal covers allow us to break the infinite level spaces up into parts that are more easily understood. In [Wei10] this breaking up is done with the help of exterior powers – more precisely: top exterior powers, i.e. determinants. For this, a construction from Hedayatzadeh's thesis [Hed10] is needed.

Theorem 3.4.5. *Let* R *be a Noetherian* $\mathcal{O}_K$*-algebra and* F *a* 1*-dimensional formal* $\mathcal{O}_K$*-module over* R *of height* $h < \infty$*. Then there exists a* 1*-dimensional formal* $\mathcal{O}_K$*-module* $\Lambda^h F$ *over* R *of height* $\binom{h}{h} = 1$ *such that, for any* $n \geq 1$*, there are* $\mathcal{O}_K$*-multilinear alternating morphisms (of formal* R*-schemes)*

$$\lambda_n \colon F^h[\pi^n] = (F[\pi^n])^h \to \Lambda^h F[\pi^n]$$

having the universal property that any $\mathcal{O}_K$*-multilinear alternating morphism from* $F^h[\pi^n]$ *to a finite locally free* $\mathcal{O}_K$*-module scheme over* R *factors through* λ_n*.*
We call $\Lambda^h F$ *the* determinant *of* F *and it is compatible with base-change.*
Also, $\pi \circ \lambda_{n+1} = \pi^h \circ \lambda_n$*, i.e., the following diagram commutes:*

$$\begin{array}{ccc} F^h[\pi^{n+1}] & \xrightarrow{\lambda_{n+1}} & \Lambda^h F[\pi^{n+1}] \\ \downarrow{\scriptstyle \pi\cdot -} & & \downarrow{\scriptstyle \pi\cdot -} \\ F^h[\pi^n] & \xrightarrow{\lambda_n} & \Lambda^h F[\pi^n] \end{array} \quad . \tag{3.8}$$

Proof. This can be found in [Hed10]: the existence of $\Lambda^h F$ is Proposition 9.2.34, the existence of λ_n along with the commutation with base-change is Proposition 9.2.33, and the commutative diagram is Remark 9.2.35. ■

Now, let us return to $\overline{F}$, our formal $\mathcal{O}_K$-module of height $h < \infty$ over k. When $\widetilde{F}$ is a lift of $\overline{F}$ over some complete Noetherian local $\mathcal{O}_K$-algebra R and S a complete I-adic R-algebra, then the diagrams 3.8, considered for every n, show us that we get an induced morphism of Tate modules

$$T(\widetilde{F})(S)^h \to T(\Lambda^h \widetilde{F})(S),$$

which is $\mathcal{O}_K$-multilinear and alternating.
Similarly we obtain $T(\widetilde{F})(S/I)^h \to T(\Lambda^h \widetilde{F})(S/I)$, and since S/I is discrete, Remark 3.4.4 tells us that after tensoring by K we get an alternating K-multilinear morphism

$$\mathrm{UC}(\widetilde{F})(S/I)^h \to \mathrm{UC}(\Lambda^h \widetilde{F})(S/I).$$

Finally, when $\mathfrak{m}_R \subset I$, we know from Proposition 3.4.2 that the universal covers evaluated at S are the same as those evaluated at S/I, such that in that case we have a K-multilinear alternating morphism

$$\mathrm{UC}(\widetilde{F})(S)^h \to \mathrm{UC}(\Lambda^h \widetilde{F})(S).$$

This construction is clearly functorial in S and therefore, when restricting both functors to the category of complete adic R-algebras with an ideal of definition containing $\mathfrak{m}_R$, we have obtained a K-multilinear alternating natural transformation[2]

$$\lambda\colon \mathrm{UC}(\widetilde{F})^h \to \mathrm{UC}(\Lambda^h \widetilde{F}). \tag{3.9}$$

This morphism is key to relating the infinite Lubin-Tate space associated to $\overline{F}$ to the infinite Lubin-Tate space associated to $\Lambda^h \overline{F}$. The latter, where the formal $\mathcal{O}_K$-module is of height 1, has already been studied in Section 3.2.

Remark 3.4.6. *Recall that both universal covers (still as functors restricted as above) are represented by $R[\![X^{1/q^\infty}]\!]$, by Proposition 3.4.3. Therefore λ corresponds to a morphism*

$$R[\![X^{1/q^\infty}]\!] \to R[\![X_1^{1/q^\infty}, \dots, X_h^{1/q^\infty}]\!].$$

Such a morphism is the same as an element $\delta(X_1, \dots, X_h)$ (the image of X) together with q^n-th roots $\delta(X_1, \dots, X_h)^{1/q^n} \in R[\![X_1^{1/q^\infty}, \dots, X_h^{1/q^\infty}]\!]$ for every $n \geq 1$.

Another important reason that the determinant is useful in relating the general case to the height 1 case is that the λ_n respect Drinfeld bases:

Proposition 3.4.7 ([Wei10][Prop. 2.5.2])**.** *If R is a complete Noetherian local $\mathcal{O}_K$-algebra, $\widetilde{F}$ a deformation of $\overline{F}$ over R, and $x_1, \dots, x_h$ a Drinfeld basis for $\widetilde{F}[\pi^n](R)$, then*

$$\lambda_n(x_1, \dots, x_h) \in \Lambda^h \widetilde{F}[\pi^n](R)$$

is also a Drinfeld basis.

[2] In fact, by the Yoneda lemma it is a morphism of formal schemes over R.

Proof. We may assume $R = A_n$, $\widetilde{F} = F_{\mathrm{univ}}$, and that the $x_i = X_i^{(n)}$ are given by the universal Drinfeld basis. We then need to prove that $\lambda_n(X_1^{(n)}, \dots, X_h^{(n)})$ is a Drinfeld basis for $\Lambda^h F_{\mathrm{univ}}[\pi^n]$, which is equivalent by Remark 3.1.2 to proving that

$$\pi_{F_{\mathrm{univ}}}^{n-1}\left(\lambda_n\left(X_1^{(n)}, \dots, X_h^{(n)}\right)\right) \overset{3.8}{=} \lambda_1\left(\pi_{F_{\mathrm{univ}}}^{n-1}\left(X_1^{(n)}\right), \dots, \pi_{F_{\mathrm{univ}}}^{n-1}\left(X_h^{(n)}\right)\right)$$

is a Drinfeld basis for $\Lambda^h F[\pi](A_n)$. From 3.7 it follows that

$$\pi_{F_{\mathrm{univ}}}^{n}\left(X_i^{(n)}\right) = X_i^{(1)}$$

when identifying A_1 with a subset of A_n via $A_1 \hookrightarrow A_n$.
Therefore, we are reduced to proving the case $n = 1$. Since $\Lambda^h F$ is height 1, it suffices by Proposition 3.2.1 to show that $\lambda_1\left(X_1^{(1)}, \dots, X_h^{(1)}\right)$ is a root of $\pi_{F_{\mathrm{univ}}}(X)/X$. That is, we must show it is non-zero.
Note that A_1 contains all roots of $\pi_{F_{\mathrm{univ}}}$: the universal Drinfeld basis in A_1 spans all the roots over $\mathcal{O}_K$ because $\pi_{F_{\mathrm{univ}}}(X)$ is a unit in $A[\![X]\!]$ times

$$\prod_{x \in (\pi^{-1}\mathcal{O}_K/\mathcal{O}_K)^h} (X - \varphi(x))$$

by Corollary 3.1.7. Therefore, the base-change $F_{\mathrm{univ}}^h[\pi]_{\widetilde{K}}$ of $F_{\mathrm{univ}}^h[\pi]$ to the field of fractions $\widetilde{K} \coloneqq \operatorname{Frac} A_1$ is represented by

$$\widetilde{K}[\![X_1, \dots, X_h]\!] / \left(\pi_{F_{\mathrm{univ}}}(X_1), \dots, \pi_{F_{\mathrm{univ}}}(X_h)\right) \cong \widetilde{K}^{hq^h}.$$

Over $\widetilde{K}$, the morphism $\lambda_1 \colon F_{\mathrm{univ}}^h[\pi]_{\widetilde{K}} \to \Lambda^h F_{\mathrm{univ}}[\pi]_{\widetilde{K}}$ corresponds to

$$\begin{aligned} \widetilde{K}[\![X]\!] / \left(\pi_{\Lambda^h F_{\mathrm{univ}}}(X)\right) &\to \widetilde{K}[\![X_1, \dots, X_h]\!] / \left(\pi_{F_{\mathrm{univ}}}(X_1), \dots, \pi_{F_{\mathrm{univ}}}(X_h)\right), \\ X &\mapsto \lambda_1(X_1, \dots, X_h). \end{aligned}$$

Now, suppose for a contradiction that $\lambda_1\left(X_1^{(1)}, \dots, X_h^{(1)}\right) = 0$, then $\lambda_1 \colon F_{\mathrm{univ}}^h[\pi](\widetilde{K}) \to \Lambda^h F_{\mathrm{univ}}[\pi](\widetilde{K})$ is the zero morphism, because the Drinfeld basis spans and λ_1 is alternating and $\mathcal{O}_K$-multilinear. Therefore, using the hq^h different projections

$$\widetilde{K}[\![X_1, \dots, X_h]\!] / \left(\pi_{F_{\mathrm{univ}}}(X_1), \dots, \pi_{F_{\mathrm{univ}}}(X_h)\right) \cong \widetilde{K}^{hq^h} \xrightarrow{\mathrm{pr}_i} \widetilde{K}$$

one sees that every $\mathrm{pr}_i\left(\lambda_1(X_1, \dots, X_h)\right)$ equals zero, whence $\lambda_1(X_1, \dots, X_h)$ itself must equal zero. This means that the morphism $\lambda_1 \colon F_{\mathrm{univ}}^h[\pi]_{\widetilde{K}} \to \Lambda^h F_{\mathrm{univ}}[\pi]_{\widetilde{K}}$ over $\widetilde{K}$ is the zero morphism. Since the determinant commutes with base-change, this morphism is the universal alternating morphism, and since it is zero, we obtain

$$\Lambda^h F_{\mathrm{univ}}[\pi]_{\widetilde{K}} = 0,$$

which contradicts that $\Lambda^h F_{\mathrm{univ}}$ is height 1. Therefore, we conclude that after all $\lambda_1\left(X_1^{(1)}, \dots, X_h^{(1)}\right) \neq 0$. ∎

Corollary 3.4.8. *For every $n \geq 0$ we have induced morphisms*

$$\mathcal{M}_{\overline{F},n} \to \mathcal{M}_{\Lambda^h \overline{F},n}.$$

Proof. All one now needs to check is that whenever two deformations $\widetilde{F}$ and $\widetilde{F}'$ of $\overline{F}$ over some complete Noetherian local $\mathcal{O}_K$-algebra R are $\star$-isomorphic, then so are $\Lambda^h \widetilde{F}$ and $\Lambda^h \widetilde{F}'$. This is trivial, because $\Lambda^h \overline{F}$ is of height 1 and thus all its deformations over R are $\star$-isomorphic. ∎

3.5 The infinite level Lubin-Tate space

We have already seen the definition of the infinite level Lubin-Tate space in the height 1 case. The general definition is almost the same:

Definition 3.5.1. *Let $\overline{F}$ be a height $h < \infty$ formal $\mathcal{O}_K$-module over k. Then the* infinite level Lubin-Tate functor associated to $\overline{F}$ *is the pointwise projective limit*

$$\mathcal{M}_{\overline{F},\infty} \coloneqq \varprojlim_n \mathcal{M}_{\overline{F},n}.$$

The infinite Lubin-Tate functor is pro-represented by the completion A of $\varinjlim_n A_n$, where the A_n are as in the previous section and the topology is the one induced by the maximal ideal $I_0 \coloneqq \mathfrak{m}_{A_0}$ of A_0 (equivalently, one can use the maximal ideal I_n of A_n for any finite n). We note that A is I_0-adically complete, because I_0 is finitely generated [Sta22, Tag 05GG] (if I_0 were not finitely generated, this would not immediately be guaranteed). Therefore, $\operatorname{Spf} A$ makes sense and

$$\mathcal{M}_{\overline{F},\infty} \cong \operatorname{Spf} A.$$

If R is any complete Noetherian local $\mathcal{O}_K$-algebra and $\widetilde{F}$ a deformation of $\overline{F}$ over R, we define an *Drinfeld level π^∞ structure on $\widetilde{F}$ over R* to be an $\mathcal{O}_K$-linear morphism

$$\varphi\colon (K/\mathcal{O}_K)^h \to \widetilde{F}[\pi^\infty](R) = \varinjlim_n \widetilde{F}[\pi^n](R)$$

such that

$$\prod_{x \in (\pi^{-1}\mathcal{O}_K/\mathcal{O}_K)^h} (X - \varphi(x))$$

divides $\pi_{\widetilde{F}}$ in $R[\![X]\!]$.
It follows that for any complete Noetherian local $\mathcal{O}_K$-algebra R

$$\mathcal{M}_{\overline{F},\infty}(R) = \left\{ \begin{array}{l} \text{isomorphism classes of pairs } (\widetilde{F}, \varphi), \\ \text{where } \widetilde{F} \text{ is a deformation of } \overline{F} \text{ and } \varphi \\ \text{a Drinfeld level } \pi^\infty \text{ structure on } \widetilde{F} \text{ over } R \end{array} \right\}, \tag{3.10}$$

where an isomorphism of pairs is defined exactly as in 3.2 (but with $n = \infty$).

Intermezzo: Drinfeld level π^∞ structures and Tate modules

Let us take an intermezzo by expanding a bit about the relation between Drinfeld level π^∞ structures and Tate modules. For this we need to generalize a bit beyond the setting we have so far been in – therefore this remark is not relevant to the rest of the text. Namely, we consider a complete local $\mathcal{O}_K$-algebra R, which is not necessarily Noetherian anymore. We notice that the subsequent objects which we may only have defined for Noetherian R have their obvious generalizations.
Let $\widetilde{F}$ be any deformation of $\overline{F}$ over R. Then we can recover the Tate module

$$T(\widetilde{F})(R) = \varprojlim_n \widetilde{F}[\pi^n](R) \subset \mathrm{UC}(\widetilde{F})(R)^3$$

[3] To evaluate the universal cover at R, we understand R to have the $\mathfrak{m}_R$-adic topology, making it complete.

from $\widetilde{F}[\pi^\infty](R)$ by

$$\begin{aligned}\mathrm{Hom}_{\mathbf{Mod}_{\mathcal{O}_K}}\left(K/\mathcal{O}_K, \widetilde{F}[\pi^\infty](R)\right) &\xrightarrow{\sim} \varprojlim_n \widetilde{F}[\pi^n](R),\\ f &\mapsto \left(f(0), f(1/\pi), f(1/\pi^2), \dots\right).\end{aligned}$$

One notices that by applying $\mathrm{Hom}_{\mathbf{Mod}_{\mathcal{O}_K}}(K/\mathcal{O}_K, -)$ some information gets lost, in general: namely, not every $x \in \widetilde{F}[\pi^\infty](R)$ sits in a compatible sequence in the Tate module. Let us try to see what we still can recover.
In general, we can only identify the Tate module tensored by K with a *subset* of the universal cover:

$$T(\widetilde{F})(R) \otimes_{\mathcal{O}_K} K = \left\{(x_0, x_1, \dots) \in \mathrm{UC}(\widetilde{F})(R) \colon x_0 \in \widetilde{F}[\pi^\infty](R)\right\}^4.$$

Therefore,

$$\begin{aligned}T(\widetilde{F})(R) \otimes_{\mathcal{O}_K} K/\mathcal{O}_K &\cong \frac{\left\{(x_0, x_1, \dots) \in \mathrm{UC}(\widetilde{F})(R) \colon x_0 \in \widetilde{F}[\pi^\infty](R)\right\}}{\left\{(x_0, x_1, \dots) \in \mathrm{UC}(\widetilde{F})(R) \colon x_0 = 0\right\}}\\ &\cong \left\{ x_0 \in \widetilde{F}[\pi^\infty](R) \colon \begin{array}{l} x_0 \text{ admits } x_i \in \widetilde{F}[\pi^\infty](R) \text{ with} \\ \pi^j_{\widetilde{F}}(x_{i+j}) = x_i \text{ for all } i, j \geq 0 \end{array} \right\}.\end{aligned}$$

The latter is generally just a subset of $\widetilde{F}[\pi^\infty](R)$. However, when R equals the ring of integers $\mathcal{O}_{\mathbf{C}}$ of the completion of the algebraic closure $\mathbf{C}$ of K for example, then we have an equality and we can retrieve $\widetilde{F}[\pi^\infty](\mathcal{O}_{\mathbf{C}})$ from the Tate module and vice versa.
Over $\mathcal{O}_{\mathbf{C}}$ (or rings of integers of field extensions of K in general) finite Drinfeld level π^n structures are *the same as* isomorphisms, and therefore Drinfeld level π^∞ structures are *the same as* isomorphisms

$$\varphi \colon (K/\mathcal{O}_K)^h \xrightarrow{\sim} \widetilde{F}[\pi^\infty](\mathcal{O}_{\mathbf{C}}).$$

Applying $\mathrm{Hom}_{\mathbf{Mod}_{\mathcal{O}_K}}(K/\mathcal{O}_K, -)$ gives an isomorphism

$$T(\widetilde{F})(\mathcal{O}_{\mathbf{C}}) \xrightarrow{\sim} \mathcal{O}_K^h$$

from which φ can again be recovered using $- \otimes_{\mathcal{O}_K} K$.
Therefore, if we use 3.10 as definition for $\mathcal{M}_{\overline{F},\infty}$ on the broader category of complete local $\mathcal{O}_K$-algebras, we see that its $\mathcal{O}_{\mathbf{C}}$-points are isomorphism classes of deformations $\widetilde{F}$ of $\overline{F}$ over $\mathcal{O}_{\mathbf{C}}$ together with a fixed isomorphism between $\mathcal{O}_K^h$ and the Tate module of $\widetilde{F}$.

Remark 3.5.2. *The reader might have found the deformation theoretic interpretation of $\mathcal{M}_{\overline{F},\infty}(R)$ as in 3.10 for a complete Noetherian local $\mathcal{O}_K$-algebra R a bit strange. Indeed, the Noetherianity condition on R seems to make it hard for deformations $\widetilde{F}$ of $\overline{F}$ over R to have many interesting π-power torsion points over R. Indeed, $\mathcal{O}_{\mathbf{C}}$, $\mathcal{O}_{K_\infty}$ and the likes are all non-Noetherian.*

[4] When R is discrete this is the same as the universal cover evaluated at R and we get Remark 3.4.4.

However, using 3.10 as definition for $\mathcal{M}_{\overline{F},\infty}$ on the broader category of complete local $\mathcal{O}_K$-algebras, this functor is not represented by A anymore... So we can have the deformation theoretic interpretation 3.10 of $\mathcal{M}_{\overline{F},\infty}$ or a geometric one, as Spf A*, but not both. We choose for the latter, as ultimately one wants to keep doing geometry. We note that points of* Spf A *in any case always correspond to specializations $\widetilde{F}$ of F_{univ} together with a Drinfeld π^∞ level structure.*

Computation of the ring A (or at least of "$A^\flat$")

We want to arrive at a description of A à la Proposition 3.2.10 in the general height case. As mentioned earlier, we will try to harness determinants and universal covers to unravel its structure, ultimately relying on what we have already seen in the height 1 case. We again follow [Wei10].

Let us begin. We let F be an arbitrary lift of $\overline{F}$ over $\mathcal{O}_K$ that we fix throughout. Viewing it as a formal $\mathcal{O}_K$-module on A, we see that

$$\mathrm{UC}(F_{\mathrm{univ}})(A) \cong \mathrm{UC}(F_{\mathrm{univ}})(A/I_0) = \mathrm{UC}(\overline{F})(A/I_0) = \mathrm{UC}(F)(A/I_0) \cong \mathrm{UC}(F)(A) \tag{3.11}$$

by Proposition 3.4.2. Recall for every n the universal Drinfeld bases $X_1^{(n)}, \dots, X_h^{(n)}$ for $F_{\mathrm{univ}}[\pi^n](A_n) \subset F_{\mathrm{univ}}[\pi^n](A)$. From 3.7 we know that in A we have $\pi_{F_{\mathrm{univ}}}(X_i^{(n+1)}) = X_i^{(n)}$, thus they define elements

$$\mathbf{X}_i \coloneqq (X_i^{(1)}, X_i^{(2)}, \dots) \in \mathrm{UC}(F_{\mathrm{univ}})(A), \text{ for } i \in \{1, \dots, h\}.$$

Via the isomorphism 3.11 we obtain corresponding elements

$$\mathbf{Z}_i = (Z_i^{(1)}, Z_i^{(2)}, \dots) \in \mathrm{UC}(F)(A),$$

where, by definition of the isomorphism, we have the formula

$$Z_i^{(n)} = \lim_{m\to\infty} \pi_F^m \left(X_i^{(m+n)} \right). \tag{3.12}$$

The h-tuple $(\mathbf{Z}_1, \dots, \mathbf{Z}_h) \in \mathrm{UC}(F)^h(A)$ corresponds, by Proposition 3.4.3, to a morphism

$$\mathcal{O}_K[\![X_1^{1/q^\infty}, \dots, X_h^{1/q^\infty}]\!] \to A, \tag{3.13}$$

and by the proofs of propositions 3.4.1 and 3.4.3 one sees that X_i^{1/q^n} is mapped to

$$\lim_{m\to\infty} \left(\widetilde{u}^{m+n} \left(Z_i^{(m+n)} \right) \right)^{q^{hm}},$$

where $\widetilde{u} \in T\mathcal{O}_K[\![T]\!]$ is a lift of the $u \in k[\![T]\!]$ such that $\pi_{\overline{F}} = \Phi^h \circ u = u \circ \Phi^h$, where $\Phi(T) = T^q$ is the q-Frobenius.
Similarly, in the height 1 case of $\Lambda^h F$, we get a morphism

$$\mathcal{O}_K[\![X^{1/q^\infty}]\!] \to \mathcal{O}_{\hat{K}_\infty}, \tag{3.14}$$

where K_∞ is defined as in Section 3.2: it is the composite of all K_n, which are the fields obtained by adding the roots of $\pi^n_{\Lambda^h F}(T)$ to K. To construct this morphism, we should choose compatible roots $X^{(n)}$ of the $\pi^n_{\Lambda^h F_{\mathrm{univ}}}(T)$ to get

$\mathbf{X} \coloneqq (X^{(1)}, X^{(2)}, \dots) \in \mathrm{UC}(\Lambda^h F_{\mathrm{univ}})(\mathcal{O}_{\hat{K}_\infty})$. Because the determinant has height 1, we may for simplicity assume that $\Lambda^h F_{\mathrm{univ}} = \Lambda^h F$, so that $\mathbf{X} \in \mathrm{UC}(\Lambda^h F)(\mathcal{O}_{\hat{K}_\infty})$. The above morphism is then defined by

$$X^{1/q^n} \mapsto \lim_{m\to\infty} \left(\widetilde{v}^{m+n}\left(X^{(m+n)}\right)\right)^{q^m},$$

where now $\widetilde{v} \in T\mathcal{O}_K[\![T]\!]$ is a lift of the v with $\pi_{\Lambda^h\overline{F}} = \Phi \circ v$.
By taking formal spectra, the just defined morphisms 3.13 and 3.14 are the same as morphisms

$$\mathcal{M}_{\overline{F},\infty} \to \mathrm{UC}(F)^h \text{ and } \mathcal{M}_{\Lambda^h\overline{F},\infty} \to \mathrm{UC}(\Lambda^h F). \tag{3.15}$$

Now, we recall the morphisms from Corollary 3.4.8

$$\mathcal{M}_{\overline{F},n} \to \mathcal{M}_{\Lambda^h\overline{F},n}$$

for every $n \geq 0$. They are compatible with the $\mathcal{M}_{\overline{F},n+1} \to \mathcal{M}_{\overline{F},n}$ and $\mathcal{M}_{\Lambda^h\overline{F},n+1} \to \mathcal{M}_{\Lambda^h\overline{F},n}$ by the commutativity of diagram 3.8. Therefore, in the limit we obtain a morphism

$$\mathcal{M}_{\overline{F},\infty} \to \mathcal{M}_{\Lambda^h\overline{F},\infty}. \tag{3.16}$$

We note that this morphism $\mathcal{O}_{\hat{K}_\infty} \to A$ restricts to

$$\mathcal{O}_{K_n} \to A_n,\ X^n \mapsto \lambda_n(X_1^{(n)}, \dots, X_h^{(n)}),$$

where $\lambda_n \colon F_{\mathrm{univ}}^h[\pi^n] \to \Lambda^h F[\pi^n]$ is the universal alternating morphism.
All in all, we have defined morphisms 3.15, 3.16, and $\lambda \colon \mathrm{UC}(F)^h \to \mathrm{UC}(\Lambda^h F)$ in 3.9. This allows us to write down the following diagram

$$\begin{array}{ccc} \mathcal{M}_{\overline{F},\infty} & \longrightarrow & \mathcal{M}_{\Lambda^h\overline{F},\infty} \\ \downarrow & & \downarrow \\ \mathrm{UC}(F)^h & \xrightarrow{\lambda} & \mathrm{UC}(\Lambda^h F) \end{array}. \tag{3.17}$$

Claim 3.5.3. *The diagram 3.17 is commutative.*

Proof. Let us denote by λ'_n the universal alternating morphism $F^h[\pi^n] \to \Lambda^h F[\pi^n]$. It suffices to check that the following diagram commutes

$$\begin{array}{ccc} A & \leftarrow & \mathcal{O}_{\hat{K}_\infty} \\ \uparrow & & \uparrow \\ \mathcal{O}_K[\![X_1^{1/q^\infty}, \dots, X_h^{1/q^\infty}]\!] & \longleftarrow & \mathcal{O}_K[\![X^{1/q^\infty}]\!] \end{array}.$$

All maps involved are $\mathcal{O}_K$-linear and continuous, so it suffices to see what happens to the X^{1/q^n}.
Let us first see what happens to X^{1/q^n} when following the curvy arrows. First, it gets sent to $\lim_{m\to\infty}\left(\widetilde{v}^{m+n}(X^{(m+n)})\right)^{q^m} \in \mathcal{O}_{\hat{K}_\infty}$ and then it ends up in A as the element

$$\lim_{m\to\infty}\left(\widetilde{v}^{m+n}\left(\lambda_{m+n}\left(X_1^{(m+n)}, \dots, X_h^{(m+n)}\right)\right)\right)^{q^h}. \tag{3.18}$$

Now let us see study the straight arrows. The vertical straight arrow corresponds to the point $\mathbf{Z} \coloneqq (\mathbf{Z}_1, \dots, \mathbf{Z}_h) \in \mathrm{UC}(F)^h(A)$. This point was obtained from $\mathbf{X} \coloneqq (\mathbf{X}_1, \dots, \mathbf{X}_h) \in \mathrm{UC}(F_{\mathrm{univ}})^h(A)$ by the isomorphisms in 3.11. These isomorphisms show that both universal covers evaluated at A can be identified with $\mathrm{UC}(\overline{F})(A/I_0)$. When we do this $\mathbf{X}$ and $\mathbf{Z}$ are identified.
Since the $\mathbf{X}_i$ lie in the Tate module $T(F_{\mathrm{univ}})(A)$, it follows that $\mathbf{X}$ and $\mathbf{Z}$ lie in $T(\overline{F})(A/I_0)^h$. Now if we recall the construction of the morphism $\lambda\colon \mathrm{UC}(F)^h \to \mathrm{UC}(\Lambda^h F)$[5], we see that the composition of the straight arrows can be identified with the point

$$\left(\lambda_1'(X_1^{(1)}, \dots, X_h^{(1)}), \lambda_2'(X_1^{(2)}, \dots, X_h^{(2)}), \dots\right) \in T(\Lambda^h \overline{F})(A/I_0).$$

Since $F_{\mathrm{univ}} \equiv F \mod I_0$ and since the determinant commutes with base-change, we have for every n that $\lambda_n \equiv \lambda_n' \mod I_0$. Therefore, the composition of straight arrows is the same point as

$$\left(\lambda_1(X_1^{(1)}, \dots, X_h^{(1)}), \lambda_2(X_1^{(2)}, \dots, X_h^{(2)}), \dots\right) \in T(\Lambda^h \overline{F})(A/I_0) \subset \mathrm{UC}(\Lambda^h F)(A).$$

This is the morphism $\mathcal{O}_K[\![X^{1/q^\infty}]\!] \to A$ that sends X^{1/q^n} to 3.18, which proves our claim. ∎

The most important theorem from [Wei10] for the understanding of the infinite level Lubin-Tate space is the following:

Theorem 3.5.4. *The diagram 3.17 is Cartesian.*[6]

In the above theorem we understand the diagram to be in the category of formal schemes, or simply in the opposite category of the category $\mathbf{CA}_{\mathcal{O}_K}$ of complete adic $\mathcal{O}_K$-algebras. In the category $\mathbf{CA}_{\mathcal{O}_K}$ the pushout of two complete adic $\mathcal{O}_K$-algebras with a finitely generated ideal of definition exists: given morphisms $S \leftarrow R \to T$ between objects $C \in \mathbf{CA}_{\mathcal{O}_K}$ that are I_C-adic, with I_S and I_T finitely generated, the pushout of this diagram is given by the *completed tensor product*

$$S \widehat{\otimes}_R T \coloneqq \varprojlim_n S \otimes_R T / \left(I_S \otimes_R T + S \otimes_R I_T\right)^n = \varprojlim_m S/I_S^m \otimes_R T/I_T^m$$

(it is $(I_S \otimes_R T + S \otimes_R I_T)$-adically complete, since this ideal is finitely generated).

Example 3.5.5. *We have already seen very simple examples of this: the completed tensor product of $R[\![X]\!]$ with itself over R equals $R[\![X, Y]\!]$ for example. The same holds with X and Y replaced by X^{1/q^∞} and Y^{1/q^∞}.*

We see that Theorem 3.5.4 is equivalent to the statement that the natural morphism

$$B \coloneqq \mathcal{O}_K[\![X_1^{1/q^\infty}, \dots, X_h^{1/q^\infty}]\!] \widehat{\otimes}_{\mathcal{O}_K[\![X^{1/q^\infty}]\!]} \mathcal{O}_{\hat{K}_\infty} \to A$$

is an isomorphism. It has the following consequences:

Corollary 3.5.6. *When K has characteristic p*

$$A \cong k[\![X_1^{1/q^\infty}, \dots, X_h^{1/q^\infty}]\!] = \mathbb{F}_q[\![X_1^{1/q^\infty}, \dots, X_h^{1/q^\infty}]\!].$$

[5] Better notation would be λ', given that we use the λ_n' for its construction rather than the λ_n.

[6] See later for a discussion of its proof.

Proof. In this case $\mathcal{O}_K = \mathbb{F}_q[\![\pi]\!]$. Consider the composition

$$\mathbb{F}_q[\![X^{1/q^\infty}]\!] \hookrightarrow \mathcal{O}_K[\![X^{1/q^\infty}]\!] \to \mathcal{O}_{\hat{K}_\infty}.$$

It sends X^{1/q^∞} to

$$\lim_{m\to\infty} \left(\widetilde{v}^{m+n}(X^{(m+n)})\right)^{q^m},$$

as we have already seen. This element has the same absolute value as $X^{(n)}$ itself (since $\widetilde{v} \in T\mathcal{O}_K[\![T]\!]$ is an invertible power series). We do the same as in Corollary 3.2.5: we give $X \in \mathbb{F}_q[\![X^{1/q^\infty}]\!]$ the absolute value as $X^{(1)} \in \mathcal{O}_{\hat{K}_\infty}$. This makes $\mathbb{F}_q[\![X^{1/q^\infty}]\!] \to \mathcal{O}_{\hat{K}_\infty}$ into an isometry. Recall from the same Corollary 3.2.5 that $\mathcal{O}_{\hat{K}_\infty} \cong \mathbb{F}_q[\![t^{1/q^\infty}]\!]$ and that in fact $|t| = |X^{(1)}|$. From this it is easily seen that the isometry is in fact an isomorphism.
Therefore, the morphism $\mathcal{O}_K[\![X^{1/q^\infty}]\!] \to \mathcal{O}_{\hat{K}_\infty}$ looks like a morphism of $\mathbb{F}_q[\![X^{1/q^\infty}]\!]$-algebras

$$\mathbb{F}_q[\![X^{1/q^\infty}]\!][\![\pi]\!] \to \mathbb{F}_q[\![X^{q^{1/q^\infty}}]\!]$$

and is therefore surjective with kernel generated by $\pi - g(X)$ for some fractional power series $g(X) \in \mathbb{F}_q[\![X^{1/q^\infty}]\!]$.
From the description of the morphism $\mathcal{O}_K[\![X^{1/q^\infty}]\!] \to \mathcal{O}_K[\![X_1^{1/q^\infty}, \dots, X_h^{1/q^\infty}]\!]$ from Remark 3.4.6 it follows that

$$\begin{aligned}
A &\cong \mathcal{O}_K[\![X_1^{1/q^\infty}, \dots, X_h^{1/q^\infty}]\!] \widehat{\otimes}_{\mathcal{O}_K[\![X^{1/q^\infty}]\!]} \mathcal{O}_{\hat{K}_\infty} \\
&\cong \mathcal{O}_K[\![X_1^{1/q^\infty}, \dots, X_h^{1/q^\infty}]\!] \widehat{\otimes}_{\mathcal{O}_K[\![X^{1/q^\infty}]\!]} \left(\mathcal{O}_K[\![X^{1/q^\infty}]\!]/\left(\pi - g(X)\right)\right) \\
&\cong \mathcal{O}_K[\![X_1^{1/q^\infty}, \dots, X_h^{1/q^\infty}]\!]/\left(\pi - g\left(\delta(X_1, \dots, X_h)\right)\right) \\
&= \mathbb{F}_q[\![X_1^{1/q^\infty}, \dots, X_h^{1/q^\infty}]\!][\![\pi]\!]/\left(\pi - g\left(\delta(X_1, \dots, X_h)\right)\right) \\
&\cong \mathbb{F}_q[\![X_1^{1/q^\infty}, \dots, X_h^{1/q^\infty}]\!].
\end{aligned}$$

■

We also get a corollary similar to Proposition 3.2.10:

Corollary 3.5.7. *Let $A^\flat$ be the inverse limit of A/π with respect to the q-Frobenius:*

$$A^\flat \coloneqq \varprojlim_{x \mapsto x^q} A/\pi.$$

Then $A^\flat \cong k[\![X_1^{1/q^\infty}, \dots, X_h^{1/q^\infty}]\!] = \mathbb{F}_q[\![X_1^{1/q^\infty}, \dots, X_h^{1/q^\infty}]\!]$.

Proof. Assume first that K is of characteristic 0. Then

$$\begin{aligned}
A/\pi &\cong \left(\mathcal{O}_K[\![X_1^{1/q^\infty}, \dots, X_h^{1/q^\infty}]\!] \widehat{\otimes}_{\mathcal{O}_K[\![X^{1/q^\infty}]\!]} \mathcal{O}_{\hat{K}_\infty}\right)/\pi \\
&\cong \mathbb{F}_q[\![X_1^{1/q^\infty}, \dots, X_h^{1/q^\infty}]\!] \widehat{\otimes}_{\mathbb{F}_q[\![X^{1/q^\infty}]\!]} \mathcal{O}_{\hat{K}_\infty}/\pi,
\end{aligned}$$

we note that the last step here is not as easy as one might expect: in principle – one can check – only A modded out by the *closure* of πA is isomorphic to $\mathbb{F}_q[\![X_1^{1/q^\infty}, \dots, X_h^{1/q^\infty}]\!] \widehat{\otimes}_{\mathbb{F}_q[\![X^{1/q^\infty}]\!]} \mathcal{O}_{\hat{K}_\infty}/\pi$; in this case, it is fine, because we will see in later that πA is closed (Lemma 3.5.19).
The map $\mathbb{F}_q[\![X^{1/q^\infty}]\!] \to \mathcal{O}_{\hat{K}_\infty}/\pi$ is surjective: even before modding out by π the map $f\colon \mathcal{O}_K[\![X^{1/q^\infty}]\!] \to \mathcal{O}_{\hat{K}_\infty}$ is surjective, since one can reconstruct the $X^{(n)}$ from

the images of the X^{1/q^m} by a limit procedure[7] reverting $f\colon X^{1/q^n} \mapsto \lim_{m\to\infty} \left(\widetilde{v}^{m+n}(X^{(m+n)})\right)^{q^m}$.
One has $|f(X^{1/q^n})| = |X^{(n)}|$, and therefore $|f(X^{q-1})| = |\pi|$, so that

$$\ker f = X^{q-1}\,\mathbb{F}_q[\![X]\!].$$

It follows that

$$A/\pi \cong \mathbb{F}_q[\![X_1^{1/q^\infty}, \ldots, X_h^{1/q^\infty}]\!]/\delta(X_1, \ldots, X_h)^{q-1}.$$

In the characteristic p case, we simply have

$$A/\pi \cong \mathbb{F}_q[\![X_1^{1/q^\infty}, \ldots, X_h^{1/q^\infty}]\!]/g\left(\delta(X_1, \ldots, X_h)\right),$$

with g as in the proof of the previous corollary.
In both cases, it is not hard to see that

$$A^\flat \cong \mathbb{F}_q[\![X_1^{1/q^\infty}, \ldots, X_h^{1/q^\infty}]\!].$$

■

Is A perfectoid?

The notation "$A^\flat$" suggests that something "perfectoid" is going on. Also suggestive is the observation that

$$A/\pi \to A/\pi,\ x \mapsto x^q$$

is surjective, which is easily deduced from Theorem 3.5.4 that

$$A \cong \mathcal{O}_K[\![X_1^{1/q^\infty}, \ldots, X_h^{1/q^\infty}]\!]\widehat{\otimes}_{\mathcal{O}_K[\![X^{1/q^\infty}]\!]}\mathcal{O}_{\hat{K}_\infty},$$

using Proposition 3.2.3 from the height 1 case.
If we pick some suitable non-zero $\varpi|\pi$ in $\mathcal{O}_{\hat{K}_\infty}$ admitting a p-th root $\varpi^{1/p}$, it follows, viewing A as an $\mathcal{O}_{\hat{K}_\infty}$-algebra in the obvious manner, that

$$A/\varpi^{1/p} \to A/\varpi, x \mapsto x^p$$

is an isomorphism (see Lemma 3.5.18 for what ϖ to pick). In the language of [Mat18][Def. 114] this means A is an "integral perfectoid" $\mathcal{O}_{\hat{K}_\infty}$-algebra. In the same text it is explained how to equip $A[1/\pi]$ with a $\hat{K}_\infty$-Banach algebra structure and it is shown [Mat18][Prop. 120] that as such $A[1/\pi]$ is a "perfectoid $\hat{K}_\infty$-algebra" as defined in [SW12][Def. 2.3.2]. This already shows that A can really be considered "perfectoid" in a way, but one problem with this is that the topology on $A[1/\pi]$ as a Banach algebra seems to induce a coarser topology on A than the I_0-adic one we have been considering so far.
There is also another way in which A can be put in the perfectoid world. Namely, one can prove that (the "adic generic fiber" of) $\mathcal{M}_{\overline{F},\infty}$ is "preperfectoid" [SW12][Thm. 6.3.4] and [Wei10][Lemma 2.10.1], which means its "adic generic fiber" is perfectoid after base-change to some perfectoid field.
For both the integral perfectoid viewpoint and the preperfectoid viewpoint one

[7] That is, $X^{(m)} = \lim_{n\to\infty} \pi_F^n \left(\widetilde{v}^{-m-n}\left(f\left(X^{1/q^{n+m}}\right)\right)\right)$.

needs the theory of adic spaces, which was developed by Huber in [Hub93] and [Hub94]. Given the rather technical nature of the theory of adic spaces and the fact that the book [SW20] gives a very good summary of the theory of adic spaces needed for perfectoid spaces already, we have decided not to expand more on the topic of perfectoid or adic spaces, nor on the two viewpoints discussed above. Another consideration contributing to this decision, is that when one is already familiar with perfectoid spaces, it is not so hard anymore to see what is going on with the infinite level Lubin-Tate space – that is, after the work we have already done. Indeed, the proof of preperfectoidness [Wei10][Lemma 2.10.1] is a rather small lemma. The main difficulty in [Wei10] lies in proving Theorem 3.5.4. Therefore, in order to show (in what way) Lubin-Tate space is perfectoid, one needs to do two rather disjoint things: do the computations that we have been the focus of this chapter and develop the theory of perfectoid spaces. Since this thesis has its focus on Lubin-Tate spaces, we are happy doing only the former.

Weinstein's proof of Theorem 3.5.4

Finally, we will present the proof of Theorem 3.5.4 given by Weinstein [Wei10][Thm. 2.7.3]. We note that Weinstein's proof is a bit less general than ours: he retricts the Lubin-Tate functors to the category of complete Noetherian local $\mathcal{O}_{\widehat{K^{\mathrm{unr}}}}$-algebras, whence there A is an $\mathcal{O}_{\widehat{K^{\mathrm{unr}}}}$-algebra with residue field $\overline{\mathbb{F}}_q$ rather than $\mathbb{F}_q$. This difference with our setting is negligible, and the proof as in [Wei10] works just as well in our case[8].
Before starting, we want to warn the reader: at the end it will become apparent that the proof has a minor gap, that is, Weinstein made a certain claim which the author was unfortunately not able to prove himself. We hope the reader will be able to find a proof of Weinstein's claim 3.5.20.

Let us now start with discussing the actual proof. It consists of multiple simple lemmas. Remember that the goal is to prove that the natural morphism

$$\varphi\colon B \to A$$

is an isomorphism. The case $h = 1$ is trivial, so we may assume $h > 1$.

The first thing [Wei10] does is study the ring A in some more detail. The maximal ideals I_0 and especially I_1, of A_0 and A_1 respectively, are of particular interest.
The $X_i^{(1)} \in I_1$ form a Drinfeld basis and thus $\pi_{F_{\mathrm{univ}}}$ divides the polynomial

$$\prod_{(a_1,\dots,a_h)\in k^h} \left(T - \left((a_1)_{F_{\mathrm{univ}}}(X_1^{(1)}) +_{F_{\mathrm{univ}}} \cdots +_{F_{\mathrm{univ}}} (a_h)_{F_{\mathrm{univ}}}(X_h^{(1)})\right)\right),$$

which is congruent modulo I_1^2 to

$$T^{q^h} - T^{q^h-1} \sum_{(a_1,\dots,a_h)\in k^h} \left(a_1 X_1^{(1)} + \cdots + a_h X_h^{(1)}\right) = T^{q^h}.$$

[8]The advantage of Weinstein's setting is that by Proposition 2.3.12 one may pick coordinates on the formal $\mathcal{O}_K$-modules such that their reductions modulo the maximal ideal become normal, which makes computations slightly easier.

Here, the last equality is clear when $p \neq 2$, and when $p = 2$ it follows from our assumption that $h > 1$ together with the fact that $2 \in I_1$.
Since the coefficient of $\pi_{F_{\text{univ}}} \mod I_1^2$ is a unit, it follows that the coefficients of the T^i for $i < q^h$ of $\pi_{F_{\text{univ}}}$ must lie in I_1^2. By Proposition 2.4.4 these coefficients generate I_0, whence we get

Lemma 3.5.8 ([Wei10][Lemma 2.8.2]). $I_0 \subset I_1^2$.

And since $\pi_{F_{\text{univ}}} \equiv \pi_F \mod I_0$, we get

Lemma 3.5.9 ([Wei10][Lemma 2.8.3]). $\pi_{F_{univ}} \equiv \pi_F \mod I_1^2$.

Here comes a lemma describing A_n/I_1 for finite n:

Lemma 3.5.10 ([Wei10][Lemma 2.8.4]). *There is an isomorphism of affine k-schemes* $\operatorname{Spec} A_n/I_1 \to \overline{F}[\pi^{n-1}]^h$.

Proof. For a Noetherian k-algebra R (with discete topology) $\operatorname{Hom}(A_n/I_1, R)$ is the set of isomorphism classes of deformations $\widetilde{F}$ of $\overline{F}$ together with a Drinfeld basis $x_1, \dots, x_h$ for $\widetilde{F}[\pi^n](R)$, such that $\widetilde{F}$ is $\star$-isomorphic to $\overline{F}$ and such that $\pi^{n-1}x_1, \dots, \pi^{n-1}x_h$ are all 0.
Thus $\operatorname{Hom}(A_n/I_1, R)$ is the set of Drinfeld bases $x_1, \dots, x_h$ for $\overline{F}[\pi^n](R)$ satisfying $\pi^{n-1}x_i = 0$. Now, any h-tuple $(x_1, \dots, x_h)$ of elements of $\overline{F}[\pi^{n-1}](R)$ is a Drinfeld basis for $\overline{F}[\pi^n](R)$: indeed, by Remark 3.1.2 this claim is equivalent to $(0, \dots, 0)$ being a Drinfeld basis for $\overline{F}[\pi](R)$, which is equivalent to T^{q^h} dividing $\pi_{\overline{F}}(T)$, which is true. ∎

Now we turn to $B = \mathcal{O}_K[\![X_1^{1/q^\infty}, \dots, X_h^{1/q^\infty}]\!] \widehat{\otimes}_{\mathcal{O}_K[\![X^{1/q^\infty}]\!]} \mathcal{O}_{\hat{K}_\infty}$. For a complete adic $\mathcal{O}_K$-algebra R with an ideal of defition containing π one has

$$\begin{aligned}
\operatorname{Hom}(B, R) &\cong \operatorname{Hom}(\mathcal{O}_K[\![X_1^{1/q^\infty}, \dots, X_h^{1/q^\infty}]\!], R) \times_{\operatorname{Hom}(\mathcal{O}_K[\![X^{1/q^\infty}]\!], R)} \operatorname{Hom}(\mathcal{O}_{\hat{K}_\infty}, R) \\
&\cong \left\{ \begin{array}{c} ((\mathbf{u}_1, \dots, \mathbf{u}_h), (t_1, t_2, \dots)) \\ \in \mathrm{UC}(F)^h(R) \times T(\Lambda^h F)(R) \end{array} : \begin{array}{l} t_n \text{ is a Drinfeld basis for } \Lambda^h F[\pi^n](R) \\ \text{and } \lambda(\mathbf{u}_1, \dots, \mathbf{u}_h) = (t_1, t_2, \dots) \end{array} \right\} \\
&\cong \left\{ \mathbf{u} \in \mathrm{UC}(F)^h(R) : \begin{array}{l} \lambda(\mathbf{u}) \text{ lies in } T(\Lambda^h F)(R) \text{ and } \forall n \text{ its } n\text{-th} \\ \text{component is a Drinfeld level } \pi^n \text{ basis} \end{array} \right\} \\
&= \left\{ \mathbf{u} \in \mathrm{UC}(F)^h(R) : \begin{array}{l} \lambda(\mathbf{u}) = \big(\lambda(\mathbf{u})^{(1)}, \lambda(\mathbf{u})^{(2)}, \dots\big) \text{ lies in } T(\Lambda^h F)(R) \\ \text{and } \lambda(\mathbf{u})^{(1)} \text{ is a Drinfeld basis for } \Lambda^h F[\pi](R) \end{array} \right\}.
\end{aligned}$$

From the proof of Claim 3.5.3 it follows that the morphism

$$\mathcal{O}_K[\![X_1^{1/q^\infty}, \dots, X_h^{1/q^\infty}]\!] \to A$$

corresponds to the h-tuple $\mathbf{Z} = (\mathbf{Z}_1, \dots, \mathbf{Z}_h) \in \mathrm{UC}(F)^h(A)$. Therefore $\varphi \in \operatorname{Hom}(B, A)$ can be identified with $\mathbf{Z}$. Furthermore, to the identity morphism in $\operatorname{Hom}(B, B)$ there correspond universal $\mathbf{Y}_i = (Y_i^{(1)}, Y_i^{(2)}, \dots) \in \mathrm{UC}(F)(B)$, for $i \in \{1, \dots, h\}$.
Let $J \subset B$ be the ideal generated by $Y_1^{(1)}, \dots, Y_h^{(1)}$ and π. This ideal is clearly topologically nilpotent. Moreover:

Lemma 3.5.11 ([Wei10][Lemma 2.8.6]). *J is an open ideal of B and φ descends to an isomorphism $B/J \to A/I$.*

Proof. We have the formula

$$Y_i^{(1)} = \lim_{m\to\infty} \pi_F^m \left(\widetilde{u}^{-m-1}(X_i^{1/q^{m+1}})\right),$$

which becomes, modulo π,

$$Y_i^{(1)} \equiv u^{-1}(X_i^{1/q}).$$

Since $\widetilde{u}^{-1}$ is an invertible power series, we get

$$J = (\pi, X_1^{1/q}, \dots, X_h^{1/q}).$$

Since B has the $(\pi, X_1, \dots, X_h)$-adic topology, it follows that J is open.
Now, let R be a complete adic $\mathcal{O}_K$-algebra with $\pi R = 0$. Then $\mathrm{Hom}(B/J, R)$ is identified with

$$\left\{ (\mathbf{u}_1, \dots, \mathbf{u}_h) \in \mathrm{UC}(F)^h(R) \colon \begin{array}{l} \mathbf{u}_i = (0, u_i^{(2)}, u_i^{(3)}, \dots) \ \forall i \in \{1, \dots, h\} \text{ and} \\ \lambda(\mathbf{u}) \text{ lies in } T(\Lambda^h F)(R) \text{ and } \forall n \text{ its } n\text{-th} \\ \text{component is a Drinfeld level } \pi^n \text{ basis} \end{array} \right\}.$$

For any $\mathbf{u} \in \mathrm{UC}(F)^h(R)$ with $u_i^{(1)}$ for all i, one has

$$\lambda(\mathbf{u})^{(1)} = \lambda_1(u_1^{(1)}, \dots, u_h^{(1)}) = 0.$$

Since $\pi R = 0$, this is automatically a Drinfeld basis, because T^q divides $\pi_{\Lambda^h \overline{F}}(T)$. Therefore,

$$\begin{aligned} \mathrm{Hom}(B/J, R) &= \left\{ (\mathbf{u}_1, \dots, \mathbf{u}_h) \in \mathrm{UC}(F)^h(R) \colon \mathbf{u}_i^{(1)} = 0 \text{ for all } i \right\} \\ &= \left\{ (\mathbf{u}_1, \dots, \mathbf{u}_h) \in T(F)^h(R) \colon \mathbf{u}_i^{(1)} = 0 \text{ for all } i \right\}. \end{aligned}$$

Let us consider the morphism $\mathrm{Hom}(A, R) \to \mathrm{Hom}(B, R) \subset \mathrm{UC}(F)^h(R)$ now. It sends a morphism $f \colon A \to R$ to the h-tuple

$$f(\mathbf{Z}) \coloneqq \left(\left(f\left(Z_i^{(1)}\right), f\left(Z_i^{(2)}\right), \dots \right) \right)_{i=1}^h = \left(\left(f\left(X_i^{(1)}\right), f\left(X_i^{(2)}\right), \dots \right) \right)_{i=1}^h =: f(\mathbf{X}),$$

where the equality is because $X_i^{(n)} \equiv Z_i^{(n)} \mod \pi$ for every n and every i. By Lemma 3.5.10, we have an isomorphism

$$\mathrm{Hom}(A/I_1, R) \cong \varprojlim_{n\geq 1} \mathrm{Hom}(A_n/I_1, R) \xrightarrow{\sim} \left\{ (\mathbf{u}_1, \dots, \mathbf{u}_h) \in T(F)^h(R) \colon \mathbf{u}_i^{(1)} = 0 \text{ for all } i \right\},$$
$$f \mapsto f(\mathbf{X}).$$

(The first isomorphism follows from $A/I_1 \cong (\varinjlim_n A_n)/I_1 (= \varinjlim_n A_n/I_n)$, which is true because I_1 is finitely generated, see [Sta22, Tag 05GG] again.)
Therefore, the composition $\mathrm{Hom}(A/I_1, R) \to \mathrm{Hom}(A, R) \to \mathrm{Hom}(B, R)$ maps $\mathrm{Hom}(A/I_1, R)$ isomorphically into $\mathrm{Hom}(B/J, R)$. We conclude by the Yoneda lemma (considering everything in the category of complete adic k-algebras). ■

Lemma 3.5.12 ([Wei10][Lemma 2.8.7])**.** *$\varphi(J)A$ is dense in I_1.*

Proof. For all i and n, we have

$$\varphi(Y_i^{(n)}) = Z_i^{(n)} = \lim_{m\to\infty} \pi_F^m(X_i^{(m+n)}),$$

by equation 3.12, where the convergence is π-adic (and thus I_1-adic).Therefore, for m large enough

$$\varphi(Y_i^{(1)}) \equiv \pi_F^{m-1}(X_i^{(m)}) \mod I_1^2.$$

By Lemma 3.5.9 it follows that

$$X_i^{(1)} \equiv \pi_{F_{\text{univ}}}^{m-1}(X_i^{(m)}) \equiv \pi_F^{m-1}(X_i^{(m)}) \equiv \varphi(Y_i^{(1)}) \mod I_1^2.$$

Thus $X_i^{(1)} \in \varphi(J)A + I_1^2$. Since the $X_i^{(1)}$ generate I_1, we get $I_1 \subset \varphi(J)A + I_1^2$, whence by iteration $I_1 \subset \varphi(J)A + I_1^N$ for every N. Since A is I_1-adically complete, the result follows. ∎

Let $\overline{A}$ denote A/π and $\overline{I}$ the image of I_1 in $\overline{A}$. Similarly, let $\overline{B} = B/\pi$ with $\overline{J} \subset \overline{B}$ the image of J. Keep writing φ for the induced morphism $\overline{B} \to \overline{A}$.
Since the q-Frobenius is surjective on $\overline{B}$, it follows from the previous lemma that it is also surjective on $\overline{A}/\overline{I}$. With some more work, it follows that:

Lemma 3.5.13 ([Wei10][Lemma 2.8.8])**.** *The q-Frobenius on $\overline{A}$ has dense image.*

Proof. By induction on $n \geq 1$ we prove that $\overline{A} = \overline{A} + \overline{I}^n$ and that $\overline{I} = (\overline{A}^q \cap \overline{I}) + \overline{I}^n$. The case $n = 1$ follows from the q-Frobenius being surjective on $\overline{A}/\overline{I}$.
Suppose both statements hold for some $n \geq 1$. By the last lemma $\varphi(\overline{J})\overline{A}$ is dense in $\overline{I}$ and thus $\overline{I} = \varphi(\overline{J})\overline{A} + \overline{I}^{n+1}$. Using the surjectivity of Frobenius on $\overline{B}$, we get $\varphi(\overline{J}) \subset \overline{A}^q \cap \overline{I}$ and we see

$$\begin{aligned}\overline{I} &\subset (\overline{A}^q \cap \overline{I})\overline{A} + \overline{I}^{n+1} \\ &= (\overline{A}^q \cap \overline{I})(\overline{A}^q + \overline{I}^n)\overline{A} + \overline{I}^{n+1} \\ &(\overline{A}^q \cap \overline{I}) + \overline{I}^{n+1} \subset \overline{I}.\end{aligned}$$

So we obtain the second statement for $n+1$: $\overline{I} = (\overline{A}^q \cap \overline{I}) + \overline{I}^{n+1}$. Plugging this in in $\overline{A} = \overline{A}^q + \overline{I}$ gives us the first statement for $n+1$. ∎

Let us introduce some more notation. We let $\overline{I}^{[q^n]}$ denote the ideal of $\overline{A}$ generated by the q^m-th powers of elements of $\overline{I}$. Since $\overline{I}$ is finitely generated, $\overline{I}^{[q^n]}$ contains $\overline{I}^N$ for N large enough. Therefore, the $\overline{I}^{[q^n]}$ for $n \geq 0$ define the topology on $\overline{A}$. This notation and insight aid the proof of the following lemma:

Lemma 3.5.14 ([Wei10][Lemma 2.8.9])**.** *The q-Frobenius is surjective on $\overline{A}$.*

Proof. Let $a \in \overline{A}$. By the previous lemma there is a $b \in \overline{A}$ such that $a - b^q \in \overline{I}^{[q]}$. That is, for some $f_i \in \overline{I}$ and $a_i \in \overline{A}$ we can write

$$a = b^q + \sum_i a_i f_i^q.$$

For these a_i we can similarly find b_i such that $a_i - b_i^q \in \overline{I}^{[q]}$ and we can find a_{ij} and $f_{ij} \in \overline{I}$ such that $a_i = b_i^q + \sum_j a_{ij} f_{ij}^q$. We can substitute this for the a_i in the above equation and going on, we find in the limit that a is the q-th power of an element in $\overline{A}$. ∎

Now, if $a \in \overline{A}$, we know by Lemma 3.5.11 that there are $b_0 \in \overline{B}$ and $a_0 \in \overline{I}$ such that $a = \varphi(b_0) + a_0$. Using the surjectivity of Frobenius on $\overline{A}$, we can again find b_1 and $a_1 \in \overline{I}$ such that

$$a = \varphi(b_0) + \varphi(b_1)^q + a_1^q.$$

We necessarily have $\varphi(b_1)^q \in \overline{I}$ and thus $b_1^q \in J$ by Lemma 3.5.11. By surjectivity of Frobenius, we can similarly write $a_1 = \varphi(b_2)^q + a_2^q$. Going on like this, we can write

$$a = \varphi(b_0) + \varphi(b_1)^q + \varphi(b_2)^{q^2} + \cdots .$$

Since every b_i is in $\overline{J}$, the sum converges before applying φ, that is:

$$a = \varphi(b_0 + b_1^q + b_2^{q^2} + \cdots).$$

Therefore:

Lemma 3.5.15 ([Wei10][Lemma 2.8.10])**.** *The induced morphism $\varphi\colon \overline{B} \to \overline{A}$ is surjective.*

As mentioned before, the I_n-adic topologies on A are the same for every n. We know give some more details:

Lemma 3.5.16 ([Wei10][Lemma 2.8.11])**.** *Let $n \geq 1$. As ideals in A_n we have $I_1 A_n \subset I_n^{[q^{h(n-1)}]} \subset I_n^{q^{h(n-1)}}$ and $I_n^{hq^{h(n-1)}} \subset I_1$.*

Proof. The last claim follows from Lemma 3.5.10. Indeed, A_n/I_1 represents $\overline{F}[\pi^{n-1}]^h$, so is isomorphic to

$$k[X_1, \ldots, X_h] / \left(\pi_{\overline{F}}^{n-1}(X_1), \ldots, \pi_{\overline{F}}^{n-1}(X_h)\right).$$

Since $\pi^{n-1}\overline{F}(X) = X^{q^{h(n-1)}} + \cdots$ it folllows that every element in the maximal ideal of this ring becomes 0 when raised to the power $q^{h(n-1)}$. Since I_n has h generators, this means that $I_n^{hq^{h(n-1)}} \subset I_1$, which is the last claim.
For the first one, we note that I_n has generators $X_1^{(n)}, \ldots, X_h^{(n)}$ and that by Lemma 3.5.9

$$(X_i^{(n)})^{q^{h(n-1)}} \equiv u^{1-n}\left(\pi_{\overline{F}}^{n-1}\left(X_i^{(n)}\right)\right) \equiv u^{1-n}\left(\pi_{F_{\text{univ}}}^{n-1}\left(X_i^{(n)}\right)\right) \equiv u^{1-n}(X_i^{(1)}) \mod I_1^2 A_n.$$

Since the $\widetilde{u}^{1-n}(X_i^{(1)})$ generate the ideal $(X_1^{(1)}, \ldots, X_h^{(1)}) = I_1$ it follows that

$$I_1 A_n \subset I_n^{[q^{h(n-1)}]} + I_1^2 A_n.$$

Iterating this containment yields

$$I_1 \subset I_n^{[q^{h(n-1)}]} + I_1^N A_n$$

for all N. The results follows as $I_1^N \subset I_n^{[q^{h(n-1)}]}$ for $N \gg 0$. ∎

At this point Weinstein needs a rather technical result from commutative algebra, about regular local rings with characteristic p residue field. The proof is just less than half a page and might be due to Weinstein himself, but we will not present it here.

Lemma 3.5.17 ([Wei10][Lemma 2.8.12])**.** *Let R be a regular local ring with maximal ideal $\mathfrak{m}$ whose residue field has characteristic p. Let Q be a power of p and let $\eta \in R$ be an element for which η^Q divides p in R. Suppose $N \geq 1$. If $x^Q \in \eta^Q R + \mathfrak{m}^{QN+1}$, then $x \in \eta R + \mathfrak{m}^{N+1}$.*

It allows him to prove

Lemma 3.5.18 ([Wei10][Lemma 2.8.13])**.** *For every $r \geq 1$, A contains an element η_r with $\eta_r^{q^r} A = \pi A$ and for which $f^{q^r} \in \pi A$ implies $f \in \eta_r A$ for all $f \in A$.*

Proof. The morphisms induced by the determinant, from Corollary 3.4.8, $\mathcal{M}_{\overline{F},r} \to \mathcal{M}_{\Lambda^h \overline{F},r}$ are the same as morphisms $\mathcal{O}_{K_r} \to A_r$. They are morphisms of $\mathcal{O}_K$-algebras, and since the $\mathcal{O}_K \to A_r$ are injective so must the $\mathcal{O}_{K_r} \to A_r$ be. Therefore, A_r contains a root $\tilde{\eta}_r$ of $\pi^r_{\Lambda^h F}(X)/\pi^{r-1}_{\Lambda^h F}(X)$, which is a degree $q^{(q-1)q^{r-1}}$ Eisenstein polynomial up to a unit in $\mathcal{O}_K[\![X]\!]$. This root thus satisfies $\tilde{\eta}_r^{(q-1)q^{r-1}} A_r = \pi A_r$. Therefore, we set $\eta_r \coloneqq \tilde{\eta}_r^{q-1}$.
Suppose now that $f^{q^r} \in \pi A$. Let $N \geq 1$ be arbitary. Let $n \geq r$ and $f_n \in A_n$ such that $f - f_n \in I_1^{hq^r N+1} A$. Then $f_n^{q^r} \in \pi A + I_1^{hq^r N+1} A$. Recall that A is the I_1-adic completion $\widetilde{A} \coloneqq \varprojlim_m A_m$. Since I_1 is finitely generated and since $\pi\widetilde{A} + I_1^{hq^r N+1}\widetilde{A}$ is an open ideal, it follows from [Sta22, Tag 05GG] that the inverse image of $\pi A + I_1^{hq^r N+1} A$ with respect to the completion $\widetilde{A} \to A$ equals $\pi\widetilde{A} + I_1^{hq^r N+1}\widetilde{A}$. Therefore, $f_n^{q^r} \in \pi\widetilde{A} + I_1^{hq^r N+1}\widetilde{A}$. So, after possibly enlarging n, we may assume that $f_n^{q^r} \in \pi A_n + I_1^{hq^r N+1} A_n$.
By Lemma 3.5.16 $f_n^{q^r} \in \pi A_n + I_n^{hq^{h(n-1)+r}N+1}$. Then by Lemma 3.5.17 $f_n \in \eta_r A_n + I_n^{hq^{h(n-1)N+1}}$. By Lemma 3.5.16, we then have $f_n \in \eta_r A_n + I_1^N A$ and thus $f \in \eta_r A + I_1^N A$ for arbitrary N. It follows that f lies in the closure of $\eta_r A$. We are done by the next lemma. ∎

Lemma 3.5.19. *Let $\eta \in A_r$. Then ηA is closed.*

Proof. If η is 0 in A (or a unit of $\mathcal{O}_{K_r}$), this is trivial. So suppose it is non-zero. Let $\ell \gg 0$ such that $\eta \notin I_q^\ell A$.
Let $N \geq 1$ and suppose $\eta x \in I_1^N A$. Choose a $\tilde{x} \in A_n$ for some $n \geq r$ such that $x - \tilde{x} \in I_1^N A$. Then, after possibly enlarging n as in the proof of the preceding lemma,

$$\eta\tilde{x} \in I_1^N A_n \subset I_n^{Nq^{h(n-1)}}.$$

Since A_n is a regular local ring, its associated graded ring $\oplus_{i\geq 0} I_n^i/I_n^{i+1}$ is isomorphic as a graded k-algebra to the polynomial ring over k (in $\dim A_n = h$ variables), see [Sta22, Tag 00NO]. Since, by Lemma 3.5.16, $\eta \notin I_n^{\ell h q^{h(n-1)}}$, we see, using the graded isomorphism with $k[T_1, \ldots, T_h]$ of the associated graded ring, that

$$\tilde{x} \in I_n^{(N-\ell h)q^{h(n-1)}} \subset I_1^{\lfloor \frac{N}{h} - \ell \rfloor} A_n,$$

where we have used Lemma 3.5.16 for the inclusion again and where we write $\lfloor \cdot \rfloor$ for the floor function. We conclude that

$$x \in I_1^{\lfloor \frac{N}{h} - \ell \rfloor} A. \tag{3.19}$$

Therefore, if $\eta x_1, \eta x_2, \ldots$ is a sequence in ηA that has a limit in A, then this limit is in ηA: indeed, let $N \gg 0$, then there is an M such that for all $i, j > M$ we

have $\eta(x_i - x_j) \in I_1^N A$. Then by 3.19 we have $x_i - x_j \in I_1^{\lfloor \frac{N}{h} - \ell \rfloor} A$, which proves the sequence $x_1, x_2, \dots$ also converges, to x, say. Then necessarily $\lim_{n\to\infty} \eta x_n = \eta x \in \eta A$. This proves closedness of ηA. ■

Weinstein did not prove this claim (with $\eta = \eta_r$), just as the claim, which we will soon see is necessary to finish, that πB is closed in B. The author regrets that he was not able to find a proof of this claim. In the above proof, we gratefully used the regularity of A_n in order to prove that ηA is closed. For B we can not do something like that, at least, not that we know of. It seems proving πB is closed requires some more knowledge of the morphism $\mathcal{O}_K[\![X^{1/q^\infty}]\!] \to \mathcal{O}_K[\![X_1^{1/q^\infty}, \dots, X_h^{1/q^\infty}]\!]$ induced by the determinant, but for this one needs to understand more details of Hedeyatzadeh's thesis [Hed10]. For this the author did not have time, unfortunately. Therefore, we hope that the reader is able to give a proof of

Unproven claim 3.5.20. *The ideal πB is closed in B.*

Let us now demonstrate how one can conclude from there.
Firstly, $\varphi\colon B \to A$ is surjective: indeed, let $a \in A$, and pick, by Lemma 3.5.15, a $b_0 \in B$ with $\varphi(b_0) + \pi a_0 = a$ for some $a_0 \in A$; we can do the same for a_0, that is, we can write $a_0 = \varphi(b_1) + \pi a_1$; continuing this process, we find

$$\varphi(b_0 + \pi b_1 + \pi^2 b_2 + \cdots) = a.$$

Now, let us turn to injectivity. Suppose $\varphi(b) = 0$. Let $N \geq 1$. Since the q-Frobenius is surjective on B/π, we can write $b \equiv c^{q^N} \mod \pi B$ for some $c \in B$. Then $\varphi(c)^{q^N} \in \pi A$, so by Lemma 3.5.18 $\varphi(c) \in \eta_N A$. By Lemma 3.5.11, $B/J \xrightarrow{\sim} A/I$, whence $c \in \eta_N B + J$, and therefore $b \in \pi B + J^{q^N}$. This means b lies in the closure of πB, which, by the unproven claim, equals πB itself. So $b \in \pi B$, but then for $d \in B$ with $\pi d = b$ we too have $\varphi(d) = 0$[9], so $d \in \pi B$ and thus $b \in \pi^B$, and so on. Thus $b = 0$, which proves injectivity and finishes the proof, modulo the unproven claim. "■"

[9]This follows from the fact that π is not a zero-divisor in A, which can be proved using that π is not a zero-divisor in $\varinjlim_n A_n$. (Use the third part of [Sta22, Tag 0315] for example.)

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